Name:

Final Exam - PHY 4604 - Fall 2012
December 13, 2012
12:30P-2:30P, NPB 1002

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

Each exam question, (a),. . ., (d), is worth 5 points, and the entire exam is out of 100 points. Some formulas are given with the relevant question.

1. **Short answer:**

(a) Write down the time dependent Schrodinger equation in one dimension.

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi$$

(b) What are the energy eigenvalues of the hydrogen atom? Make sure to specify the allowed values of the quantum number(s).

$$E_n = -\frac{13.6\,eV}{n^2} \quad \text{for } n = 1, 2, 3, \ldots$$

(c) Write down the wave function for two identical Bosons where one particle is in the state $\psi_a(r)$ and the other is in state $\psi_b(r)$.

$$\psi(r_1, r_2) = \frac{1}{\sqrt{2}}\left(\psi_a(r_1)\psi_b(r_2) + \psi_b(r_1)\psi_a(r_2)\right)$$

(d) State the uncertainty principle for position and momentum.

$\Delta x \Delta p \geq \frac{\hbar}{2}$, where

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$$

2. **One Dimensional Schrodinger Equation:**
Consider the potential

$$\begin{aligned} V(x) &= 0 \text{ for } x < -a \\ V(x) &= -V_0 \text{ for } -a < x < a \\ V(x) &= 0 \text{ for } a < x, \end{aligned}$$

where $-V_0 < 0$. In the following take the energy to be less than zero, $E < 0$, and consider an *odd* wave function, $\psi(-x) = -\psi(x)$.

(a) What is the form of the physical solution to the time independent Schrodinger equation for $x > a$? Make sure to specify any wave vectors or other constants.

$$\psi(x) = A e^{-\rho x}, \text{ where } \rho = \sqrt{\frac{2m|E|}{\hbar^2}}.$$

(b) What is the form of the *odd* solution in the region $-a < x < a$? Again make sure to specify any wave vectors or other constants.

$$\psi(x) = B \sin kx, \text{ where } k = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

(c) Apply the boundary conditions at $x = a$ to derive the condition for a bound state solution.

$$\psi(a) = Ae^{-\rho a} = B\sin ka$$

$$\psi'(a) = -\rho A e^{-\rho a} = kB\cos ka$$

$$\rightarrow \tan(ka) = -\frac{k}{\rho}, \text{ where } k = \sqrt{2m(E+V_0)/\hbar^2}.$$

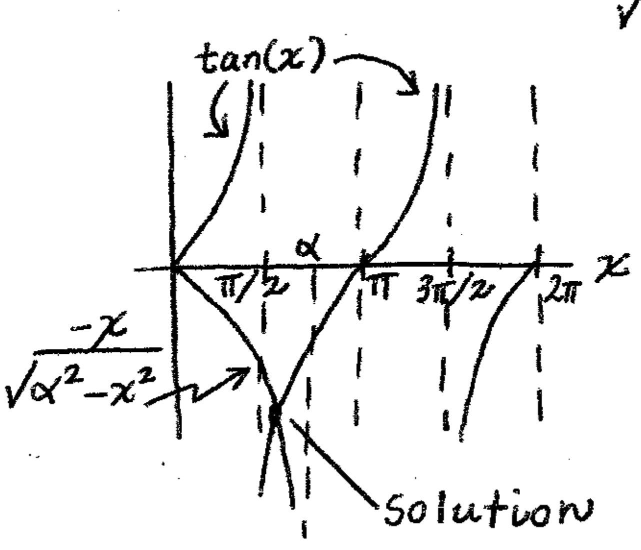
$$\rho = \sqrt{2m(-E)/\hbar^2}$$

Note $k^2 + \rho^2 = 2mV_0/\hbar^2$.

(d) Solve the bound state condition graphically. Is there always an odd bound state? If not, what is the condition that there be at least one odd bound state solution?

Let $x = ka$. $0 \leq x \leq \sqrt{2mV_0/\hbar^2}\, a \equiv \alpha$

$$\rightarrow \tan(x) = \frac{-x}{\sqrt{\alpha^2 - x^2}}$$



In order for there to be a least one solution

$$\alpha > \frac{\pi}{2}$$

$$\rightarrow \frac{2mV_0 a^2}{\hbar^2} > \frac{\pi^2}{4}$$

3. **Harmonic Oscillator:**
The Hamiltonian of the one-dimensional harmonic oscillator is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2.$$

Suppose state of system at $t = 0$ is

$$\frac{1}{\sqrt{3}}\psi_1(x) + i\sqrt{\frac{2}{3}}\psi_2(x),$$

where $\psi_n(x)$ for $n = 0, 1, 2, \ldots$ are the energy eigenstates of the Hamiltonian.

(a) What is the wave function, $\psi(x,t)$, at time $t > 0$?

$$\psi(x,t) = \frac{1}{\sqrt{3}}\psi_1(x)e^{-i\frac{3}{2}\omega t} + i\sqrt{\frac{2}{3}}\psi_2(x)e^{-i\frac{5}{2}\omega t}$$

because $E_n = \hbar\omega(n+\frac{1}{2}) = \frac{\hbar\omega}{2}, \frac{3\hbar\omega}{2}, \frac{5\hbar\omega}{2}, \ldots$

(b) Compute the expectation values of x and x^2 at time t, where

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-).$$

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{1}{\sqrt{3}}\langle 1|e^{i\frac{3}{2}\omega t} - i\sqrt{\frac{2}{3}}e^{i\frac{5}{2}\omega t}\langle 2|\right)(a+a^\dagger)$$
$$\left(\frac{1}{\sqrt{3}}|1\rangle e^{-i\frac{3}{2}\omega t} + i\sqrt{\frac{2}{3}}e^{-i\frac{5}{2}\omega t}|2\rangle\right)$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\,\frac{2i}{3}\left(-e^{i\omega t} + e^{-i\omega t}\right) = \boxed{\frac{4}{3}\sqrt{\frac{\hbar}{2m\omega}}\sin(\omega t)}$$

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\left(\frac{1}{3}(2\cdot 1+1) + \frac{2}{3}(2\cdot 2+1)\right) = \boxed{\frac{\hbar}{2m\omega}\frac{13}{3}}$$

because $(a_+ + a_-)^2 = a_+^2 + a_-^2 + a_+a_- + a_-a_+$

and $\langle n|a_+a_- + a_-a_+|n\rangle = 2n+1$.

(c) If an energy measurement is made on the $\psi(x,t)$ from part (a), what are the possible outcomes and their associated probabilities?

outcome	probability
$\frac{3\hbar\omega}{2}$	$\frac{1}{3}$
$\frac{5\hbar\omega}{2}$	$\frac{2}{3}$

(d) Construct a state which has an energy expectation value of $2\hbar\omega$ and also has a position expectation value of zero independent of time. (There is more that one such state – just list one of them).

$2\hbar\omega$ is between $\frac{3}{2}\hbar\omega$ $(n=1)$ and $\frac{5}{2}\hbar\omega$ $(n=2)$. However, $\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$ will have time dependence for $\langle x\rangle$. The states must differ in n by more than 1. For example try

$$|\psi\rangle = \alpha|0\rangle + \beta|2\rangle \quad , \alpha^2+\beta^2=1$$

$$\langle\psi|H|\psi\rangle = \alpha^2\frac{\hbar\omega}{2} + \beta^2\frac{5}{2}\hbar\omega = 2\hbar\omega$$

$$\rightarrow \frac{\alpha^2}{2} + (1-\alpha^2)\frac{5}{2} = 2 \rightarrow \frac{1}{2} = 2\alpha^2 \rightarrow \alpha^2 = \frac{1}{4}$$

$$\beta^2 = 3/4$$

$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|2\rangle$ works.

4. **Formalism:**
The Pauli spin matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the following we consider the Hamiltonian for a spin 1/2 particle with $H = \mu_B B \sigma_z$.

(a) At $t = 0$ the spin is in the $-x$ direction, which means that it is in the eigenstate of S_x with eigenvalue $-\hbar/2$. What is column vector for $|\psi(0)\rangle$?

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(b) At time $t > 0$ what is $|\psi(t)\rangle$?

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\mu_B B t/\hbar} \\ -e^{+i\mu_B B t/\hbar} \end{pmatrix}$$

(c) An S_y measurement is performed at time t. What are the possible outcomes and their associated probabilities?

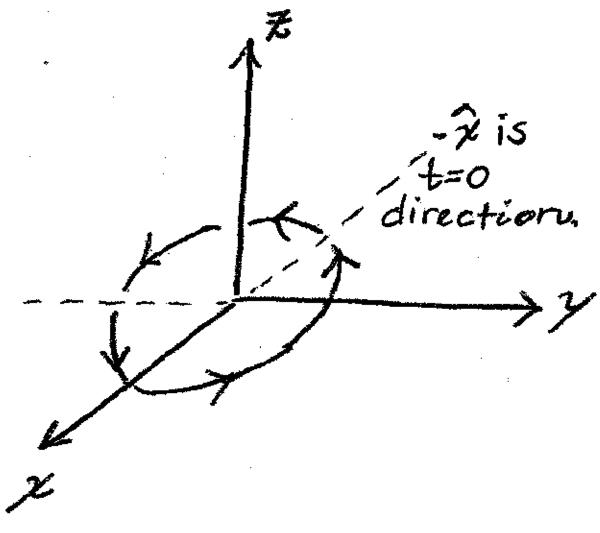
outcome	S_y eigenvec.	Probability
$-\hbar/2$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$	$\left\|\frac{1}{\sqrt{2}}(1\ \ i)\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\varphi}\\-e^{i\varphi}\end{pmatrix}\right\|^2=\left\|\frac{e^{-i\varphi}-ie^{i\varphi}}{2}\right\|^2$
$+\hbar/2$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$	$\left\|\frac{1}{\sqrt{2}}(1\ \ -i)\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\varphi}\\-e^{i\varphi}\end{pmatrix}\right\|^2=\left\|\frac{e^{-i\varphi}+ie^{i\varphi}}{2}\right\|^2$

Here $\varphi=\mu_B Bt/\hbar$ and

$$\left|\frac{e^{-i\varphi}\pm ie^{i\varphi}}{2}\right|^2=\frac{1}{4}\left(1+1\pm i\left(e^{2i\varphi}-e^{-i2\varphi}\right)\right)$$
$$=\frac{1}{2}\left(1\mp\sin(2\varphi)\right)$$

$$\Rightarrow \boxed{\text{Prob.}\left(\frac{\hbar}{2}\right)=\frac{1}{2}\left(1-\sin\left(\frac{\mu_B Bt}{\hbar}\right)\right),\ \ \text{Prob.}\left(-\frac{\hbar}{2}\right)=\frac{1}{2}\left(1+\sin\left(\frac{\mu_B Bt}{\hbar}\right)\right)}$$

(d) Interpret your result of part (c) by sketching how the spin is rotating in three dimensions.



Note that $\langle S_y\rangle=$

$$=\frac{\hbar}{2}\cdot\frac{1}{2}\left(1-\sin\left(\frac{\mu_B Bt}{\hbar}\right)\right)$$
$$-\frac{\hbar}{2}\cdot\frac{1}{2}\left(1+\sin\left(\frac{\mu_B Bt}{\hbar}\right)\right)$$
$$=-\frac{\hbar}{2}\sin\left(\frac{\mu_B Bt}{\hbar}\right)$$

5. **Angular momentum:**

(a) Compute the commutator $[L_-, (L_+)^2]$?

$$[L_-, L_+] = [L_x - iL_y, L_x + iL_y]$$
$$= i[L_x, L_y] - i[L_y, L_x] = -2\hbar L_z$$

$$[L_-, (L_+)^2] = [L_-, L_+]L_+ + L_+[L_-, L_+]$$
$$= -2\hbar(L_z L_+ + L_+ L_z)$$

(b) For $j = 2$ and $m = 1$ what is the expectation value $\langle j, m|(J_x)^2 + (J_y)^2|j, m\rangle$?

$$J^2 = J_x^2 + J_y^2 + J_z^2 \rightarrow J_x^2 + J_y^2 = J^2 - J_z^2$$

$$\Rightarrow \langle j, m| J_x^2 + J_y^2 |j, m\rangle = \hbar^2 j(j+1) - \hbar^2 m^2.$$
$$= \hbar^2(2(2+1) - 1^2)$$
$$= 5\hbar^2$$

(c) For the Hydrogen atom, how many states are there with an energy of $E = -13.6eV/4$ and z-component of the angular momentum, L_z, of zero?

These states have $n=2$ so $l=0$ or $l=1$.

Both $l=0$ & $l=1$ can have $m=0$.

$\rightarrow$ 2 states

(d) Determine the two missing elements in the table of Clebsh-Gordon coefficients for $j_1 = 3/2$, $j_2 = 1/2$ shown below. Write down in Dirac notation an expression for the $j = 1$, $m = 1$ state.

3/2×1/2		2		
		+2	2	1
+3/2	+1/2	1	+1	+1
+3/2	−1/2		1/4	?
+1/2	+1/2		3/4	?

3/4

−1/4

From the table:

$$|j=2, m=1\rangle = \frac{1}{\sqrt{4}} \left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle \otimes \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{3}{4}} \left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle \otimes \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle$$

j_1 m_1 j_2 m_2 j_1 m_1 j_2 m_2

$|j=1, m=1\rangle$ is orthogonal to this. $\rightarrow$

$$\rightarrow |j=1, m=1\rangle = \sqrt{\tfrac{3}{4}} \left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle \otimes \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle - \frac{1}{\sqrt{4}} \left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle \otimes \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle$$