Formulas to Memorize for Exam 2

The following is a list of formulas and relations that will be used in the short answer section. For this exam there is a short answer section and then four problems: one taken from each of the homework assignments since the last exam. There will be other things you need to know, e.g. how to diagonalize matrices, that are not listed below.

Properties of the inner product:

$$\langle f|g \rangle = \int dx\, f^*(x) g(x)$$
$$\langle f|g_1 + g_2 \rangle = \langle f|g_1 \rangle + \langle f|g_2 \rangle$$
$$\langle f_1 + f_2|g \rangle = \langle f_1|g \rangle + \langle f_2|g \rangle$$
$$\langle f|cg \rangle = c\langle f|g \rangle$$
$$\langle cf|g \rangle = c^*\langle f|g \rangle$$
$$\langle f|g \rangle^* = \langle g|f \rangle$$

Completeness:

$$|\psi\rangle = \sum_n c_n |\psi_n\rangle$$
$$c_n = \langle \psi_n|\psi \rangle$$
$$|\psi\rangle = \sum_n |\psi_n\rangle\langle \psi_n|\psi \rangle$$
$$1 = \sum_n |\psi_n\rangle\langle \psi_n|$$

Here the $|\psi_n\rangle$ are any complete set of orthonormal states. They could be energy eigenstates, but they also be eigenstates of another Hermitian operator. $|\psi\rangle$ is an arbitrary state.

Matrix representation: Matrix representation. With $c_n = \langle \psi_n|\psi \rangle$ above, the state $|\psi\rangle$ is represented as column vector

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix},$$

and the corresponding ket $\langle \psi|$ is represented as a row vector

$$\begin{pmatrix} c_1^* & c_2^* & \cdots \end{pmatrix}.$$

A linear operator A is represented as a matrix:

$$\begin{pmatrix} \langle \psi_1|A|\psi_1 \rangle & \langle \psi_1|A|\psi_2 \rangle & \cdots \\ \langle \psi_2|A|\psi_1 \rangle & \langle \psi_2|A|\psi_2 \rangle & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Linear operators, Q, satisfy the property

$$Q(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) = \alpha Q|\psi_1\rangle + \beta Q|\psi_2\rangle.$$

Hermitian operators: The adjoint of an operator Q is denoted as $Q^\dagger$ and defined by

$$\langle f|Qg\rangle = \langle Q^\dagger f|g\rangle.$$

The adjoint satisfies

$$\begin{aligned}
(cQ)^\dagger &= c^* Q^\dagger \\
(AB)^\dagger &= B^\dagger A^\dagger.
\end{aligned}$$

In matrix form the adjoint is the complex conjugate transpose. A Hermitian operator is one for which $Q = Q^\dagger$. Hermitian operators have real eigenvalues, and eigenvectors with different eigenvalues are orthogonal.

Uncertainty principle If two operators, A and B, commute, then one can find a complete set of simultaneous eigenvectors of A and B, i.e., states which are both eigenvectors of A and B, but with different eigenvalues in general. The converse is also true: If a complete set of simultaneous eigenvectors of A and B exists, then the operators A and B commute. If A and B do not commute, then one can not fine a complete set of simultaneous eigenvectors. The generalized uncertainty relation is

$$\sigma_A \sigma_B \geq |\frac{1}{2i}\langle [A, B]\rangle|.$$

Postulates of quantum mechanics: These are given in the lecture notes.

Position and momentum eigenvectors: The position eigenvectors are $g_y(x) = \delta(x - y)$, and the momentum eigenvectors are $f_p(x) = e^{ipx/\hbar}/\sqrt{2\pi\hbar}$. These are not normalizable in the usual sense where the integral of the magnitude squared is one, but there are still normalization conditions:

$$\begin{aligned}
\langle f_p|f_{p'}\rangle &= \delta(p - p') \\
\langle g_y|g_{y'}\rangle &= \delta(y - y').
\end{aligned}$$

Angular momentum: The commutators of the angular momentum operators are

$$\begin{aligned}
[J_x, J_y] &= i\hbar J_z \\
[J_y, J_z] &= i\hbar J_x \\
[J_z, J_x] &= i\hbar J_y.
\end{aligned}$$

The angular momentum eigenvectors $|j, m\rangle$ with $j = 0, 1/2, 1, 3/2, 2, \ldots$ and $m = -j, (-j+1), \ldots, j$ satisfy

$$\begin{aligned}
J^2|j, m\rangle &= \hbar^2 j(j+1)|j, m\rangle \\
J_\pm &= J_x \pm iJ_y \\
J_+|j, m\rangle &= \hbar\sqrt{j(j+1) - m(m+1)}|j, m+1\rangle \\
J_-|j, m\rangle &= \hbar\sqrt{j(j+1) - m(m-1)}|j, m-1\rangle \\
J_z|j, m\rangle &= \hbar m|j, m\rangle.
\end{aligned}$$

These relations can be used to compute the matrix elements of the angular momentum operator. For the orbital part of the angular momentum j must be an integer.

Spherically symmetric problems: The Hamiltonian in three dimensions is

$$H = \frac{p^2}{2m} + V = -\frac{\hbar^2}{2m}\nabla^2 + V(r).$$

The solution to the Schrodinger equation for a spherically symmetric problem in 3D has the form $\psi(r, \theta, \phi) = R(r)Y_{l,m}(\theta, \phi)$, where $R(r) = u(r)/r$ and

$$\frac{-\hbar^2}{2m}\frac{d^2u}{dx^2} + \frac{\hbar^2 l(l+1)}{2mr^2}u(r) + V(r)u(r) = Eu(r).$$

The normalization condition on the u is

$$1 = \int_0^\infty |u(r)|^2 dr = \int_0^\infty r^2 |R(r)|^2 dr.$$

For small r, $u(r) \propto r^{l+1}$ and $R(r) \propto r^l$. The bound state energies for the hydrogen atom are $E = -13.6eV/n^2$, where $n = 1, 2, \ldots$ and $l = 0, 1, \ldots, n-1$.

Formulas Printed on Exam 2

If needed for a particular problem, you will be given tables of the spherical harmonics, the radial functions for the hydrogen atom, and the spherical Bessel functions.