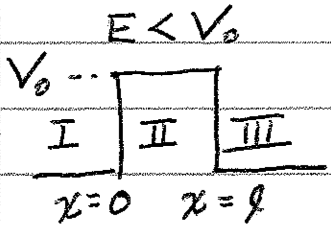
Potential Barrier:



$$k = \sqrt{\frac{2mE}{\hbar^2}}, \quad \rho = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

I. $\varphi(x) = A_1 e^{ikx} + A_1' e^{-ikx}$

II. $\varphi(x) = B_2 e^{\rho x} + B_2' e^{-\rho x}$

III. $\varphi(x) = A_3 e^{ikx} + A_3' e^{-ikx}$

Boundary conditions:

$$\varphi(0^-) = \varphi(0^+) \rightarrow A_1 + A_1' = B_2 + B_2'$$

$$\varphi'(0^-) = \varphi'(0^+) \rightarrow ik\,(A_1 - A_1') = \rho\,(B_2 - B_2')$$

$$\varphi(l^-) = \varphi(l^+) \rightarrow A_3 e^{ikl} + A_3' e^{-ikl} = B_2 e^{\rho l} + B_2' e^{-\rho l}$$

$$\varphi'(l^-) = \varphi'(l^+) \rightarrow ik\,(A_3 e^{ikl} - A_3' e^{-ikl})$$
$$= \rho\,(B_2 e^{\rho l} - B_2' e^{-\rho l})$$

Matrix notation:

$$\begin{pmatrix} 1 & 1 \\ ik_1 & -ik_1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_1' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \rho & -\rho \end{pmatrix} \begin{pmatrix} B_2 \\ B_2' \end{pmatrix}$$

$$\begin{pmatrix} e^{\rho l} & e^{-\rho l} \\ \rho e^{\rho l} & -\rho e^{-\rho l} \end{pmatrix} \begin{pmatrix} B_2 \\ B_2' \end{pmatrix} = \begin{pmatrix} e^{ikl} & e^{-ikl} \\ ike^{ikl} & -ike^{-ikl} \end{pmatrix} \begin{pmatrix} A_3 \\ A_3' \end{pmatrix}$$

<u>Inverse of a 2×2 matrix:</u>

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} A_1 \\ A_1' \end{pmatrix} = \frac{1}{-2ik_1}\begin{pmatrix} -ik & -1 \\ -ik & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ \rho & -\rho \end{pmatrix}\begin{pmatrix} B_2 \\ B_2' \end{pmatrix}$$

$$= \frac{1}{-2ik_1}\begin{pmatrix} -ik-\rho & -ik+\rho \\ -ik+\rho & -ik-\rho \end{pmatrix}\begin{pmatrix} B_2 \\ B_2' \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(1-i\frac{\rho}{k}) & \frac{1}{2}(1+i\frac{\rho}{k}) \\ \frac{1}{2}(1+i\frac{\rho}{k}) & \frac{1}{2}(1-i\frac{\rho}{k}) \end{pmatrix}\begin{pmatrix} B_2 \\ B_2' \end{pmatrix}$$

$$\begin{pmatrix} B_2 \\ B_2' \end{pmatrix} = \frac{1}{-2\rho}\begin{pmatrix} -\rho e^{-\rho \ell} & -e^{-\rho \ell} \\ -\rho e^{\rho \ell} & e^{\rho \ell} \end{pmatrix}\begin{pmatrix} e^{ik\ell} & e^{-ik\ell} \\ ik e^{ik\ell} & -ik e^{-ik\ell} \end{pmatrix}\begin{pmatrix} A_3 \\ A_3' \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(1+\frac{ik}{\rho})e^{-\rho \ell}e^{ik\ell} & \frac{1}{2}(1-\frac{ik}{\rho})e^{-\rho \ell}e^{-ik\ell} \\ \frac{1}{2}(1-\frac{ik}{\rho})e^{\rho \ell}e^{ik\ell} & \frac{1}{2}(1+\frac{ik}{\rho})e^{\rho \ell}e^{-ik\ell} \end{pmatrix}\begin{pmatrix} A_3 \\ A_3' \end{pmatrix}$$

$1 + \frac{ik}{\rho} = \frac{\rho + ik}{\rho} = \frac{z}{\rho}$, where $z = \rho + ik$

$1 - \frac{i\rho}{k} = \frac{ik+\rho}{ik} = \frac{z}{ik}$

$$\rightarrow \begin{pmatrix} A_1 \\ A_1' \end{pmatrix} = \frac{1}{2ik} \begin{pmatrix} z & -z^* \\ -z^* & z \end{pmatrix} \begin{pmatrix} B_2 \\ B_2' \end{pmatrix}$$

$$\begin{pmatrix} B_2 \\ B_2' \end{pmatrix} = \frac{1}{2\rho} \begin{pmatrix} z e^{-\rho l} e^{ikl} & z^* e^{-\rho l} e^{-ikl} \\ z^* e^{\rho l} e^{ikl} & z\, e^{\rho l} e^{-ikl} \end{pmatrix} \begin{pmatrix} A_3 \\ A_3' \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} A_1 \\ A_1' \end{pmatrix} = \frac{1}{4i\rho k} \begin{pmatrix} z^2 e^{-\rho l} e^{ikl} - z^{*2} e^{\rho l} e^{ikl} & |z|^2 e^{-\rho l} e^{-ikl} - |z|^2 e^{\rho l} e^{-ikl} \\ -|z|^2 e^{-\rho l} e^{ikl} + |z|^2 e^{\rho l} e^{ikl} & -z^{*2} e^{-\rho l} e^{-ikl} + z^2 e^{\rho l} e^{-ikl} \end{pmatrix} \begin{pmatrix} A_3 \\ A_3' \end{pmatrix}$$

$$|z|^2 = \rho^2 + k^2$$
$$z^2 = \rho^2 - k^2 + 2ik\rho$$
$$z^{*2} = \rho^2 - k^2 - 2ik\rho$$

$$\rightarrow \begin{pmatrix} A_1 \\ A_1' \end{pmatrix} = \frac{1}{4i\rho k} \begin{pmatrix} (\rho^2 - k^2)(e^{-\rho l} - e^{\rho l}) e^{ikl} + 2ik\rho (e^{-\rho l} + e^{\rho l}) e^{ikl} & (\rho^2 + k^2)(e^{-\rho l} - e^{\rho l}) e^{-ikl} \\ (\rho^2 + k^2)(e^{\rho l} - e^{-\rho l}) e^{ikl} & (\rho^2 - k^2)(e^{\rho l} - e^{-\rho l}) e^{-ikl} + 2ik\rho (e^{\rho l} + e^{-\rho l}) e^{-ikl} \end{pmatrix} \begin{pmatrix} A_3 \\ A_3' \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(\rho l) e^{ikl} + \frac{k^2 - \rho^2}{2i\rho k} \sinh(\rho l) e^{ikl} & -\frac{\rho^2 + k^2}{2i\rho k} \sinh(\rho l) e^{-ikl} \\ \frac{\rho^2 + k^2}{2i\rho k} \sinh(\rho l) e^{ikl} & \cosh(\rho l) e^{-ikl} + \frac{\rho^2 - k^2}{2i\rho k} \sinh(kl) e^{-ikl} \end{pmatrix} \begin{pmatrix} A_3 \\ A_3' \end{pmatrix}$$

<u>Continue to simplify:</u>

$$\begin{pmatrix} A_1 \\ A_1' \end{pmatrix} = \begin{pmatrix} \cosh(\rho l) + \frac{k^2-\rho^2}{2i\rho k}\sinh(\rho l) & -\frac{\rho^2+k^2}{2i\rho k}\sinh(\rho l) \\ \frac{\rho^2+k^2}{2i\rho k}\sinh(\rho l) & \cosh(\rho l) + \frac{\rho^2-k^2}{2i\rho k}\sinh(kl) \end{pmatrix} \begin{pmatrix} A_3 e^{ikl} \\ A_3' e^{-ikl} \end{pmatrix}$$

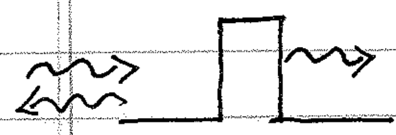
This matrix hand determinant:

$$\det = \cosh^2(\rho l) - \left(\frac{k^2-\rho^2}{2i\rho k}\right)^2 \sinh^2(\rho l) + \left(\frac{\rho^2+k^2}{2i\rho k}\right)^2 \sinh^2(\rho l)$$

$$= \cosh^2(\rho l) - \frac{1}{4\rho^2 k^2}\left((\rho^2+k^2)^2 - (k^2-\rho^2)^2\right) \times \sinh^2(\rho l)$$

$$= \cosh^2(\rho l) - \frac{4\rho^2 k^2}{4\rho^2 k^2}\sinh^2(\rho l) = \underline{1}.$$

$$\rightarrow \begin{pmatrix} A_3 e^{ikl} \\ A_3' e^{-ikl} \end{pmatrix} = \begin{pmatrix} \cosh(\rho l) + \frac{\rho^2-k^2}{2i\rho k}\sinh(kl) & \frac{\rho^2+k^2}{2i\rho k}\sinh(\rho l) \\ -\frac{\rho^2+k^2}{2i\rho k}\sinh(\rho l) & \cosh(\rho l) + \frac{k^2-\rho^2}{2i\rho k}\sinh(kl) \end{pmatrix} \begin{pmatrix} A_1 \\ A_1' \end{pmatrix}$$



$A_3' = 0$

$$\Rightarrow \frac{A_1}{A_3 e^{ik\ell}} = \cosh(\rho\ell) + \frac{k^2-\rho^2}{2i\rho k}\sinh(\rho\ell)$$

$$\frac{A_1'}{A_3 e^{ik\ell}} = \frac{\rho^2+k^2}{2i\rho k}\sinh(\rho\ell)$$

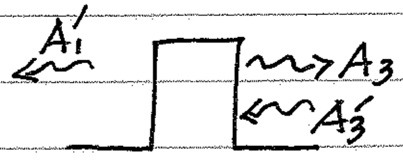
$$T = \frac{k|A_3|^2}{k|A_1|^2} = \frac{1}{\cosh^2(\rho\ell) + \frac{(k^2-\rho^2)^2}{4\rho^2k^2}\sinh^2(\rho\ell)}$$

$$R = \frac{k|A_1'|^2}{k|A_1|^2} = \frac{\frac{(\rho^2+k^2)^2}{4\rho^2k^2}\sinh^2(\rho\ell)}{\cosh^2(\rho\ell) + \frac{(k^2-\rho^2)^2}{4\rho^2k^2}\sinh^2(\rho\ell)}$$

Because $\cosh^2(\rho\ell) - \sinh^2(\rho\ell) = 1$,
$\cosh^2(\rho\ell) = 1 + \sinh^2(\rho\ell)$, and

$$\boxed{\begin{aligned} T &= \frac{1}{1 + \frac{(k^2+\rho^2)^2}{4\rho^2k^2}\sinh^2(\rho\ell)} \\ R &= \frac{\frac{(k^2+\rho^2)^2}{4\rho^2k^2}\sinh^2(\rho\ell)}{1 + \frac{(k^2+\rho^2)^2}{4\rho^2k^2}\sinh^2(\rho\ell)} \end{aligned}}$$

$\rightarrow T + R = 1$ ✓



$A_1 = 0$

$$\frac{A_3 e^{ik\ell}}{A_1'} = \frac{\rho^2+k^2}{2i\rho k}\sinh(\rho\ell)$$

$$\frac{A_3' e^{-ik\ell}}{A_1'} = \cosh(\rho\ell) + \frac{k^2-\rho^2}{2i\rho k}\sinh(\rho\ell)$$

$$\rightarrow \quad T = \frac{k|A_1'|^2}{k|A_3'|^2} = \frac{1}{\cosh^2(\rho\ell) + \frac{(k^2-\rho^2)^2}{4\rho^2k^2}\sinh^2(\rho\ell)}$$

$$R = \frac{k|A_3|^2}{k|A_3'|^2} = \frac{\frac{(\rho^2+k^2)}{4\rho^2k^2}\sinh^2(\rho\ell)}{\cosh^2(\rho\ell) + \frac{(k^2-\rho^2)^2}{4\rho^2k^2}\sinh^2(\rho\ell)}$$

<u>These are the same as the first case.</u>

For $\rho l \gg 1$, $\sinh(\rho l) \approx \frac{1}{2} e^{\rho l}$ and

$$\left.\begin{aligned} T &\approx \frac{16\rho^2 k^2}{(\rho^2+k^2)^2} e^{-2\rho l} \\ R &\approx 1. \end{aligned}\right\} \text{quantum tunneling}$$