Name:

Exam 1 - PHY 4604 - Fall 2013
October 3, 2013
8:20-10:20PM, NPB 1002

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

Unless otherwise noted, all parts (a), (b), ... are worth 5 points, and the entire exam is 100 points.

Harmonic oscillator:

$$
\begin{aligned}
a_+ &= \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x) \\
a_- &= \frac{1}{\sqrt{2\hbar m\omega}}(+ip + m\omega x) \\
x &= \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \\
p &= i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-) \\
\psi_0(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\
\psi_n &= \frac{1}{\sqrt{n!}}\left(a_+\right)^n \psi_0
\end{aligned}
$$

Delta function potential $V(x) = \alpha\, \delta(x)$:

$$
\frac{d\varphi(0^+)}{dx} - \frac{d\varphi(0^-)}{dx} = \frac{2m\alpha}{\hbar^2}\varphi(0).
$$

1. **Short answer section**

(a) Write down the time *dependent* Schrodinger equation in one dimension.

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi$$

(b) What are the energy eigenvalues for the harmonic oscillator?

$$E = \hbar\omega\left(n+\frac{1}{2}\right) \text{ for } n = 0, 1, 2, \ldots$$

(c) What does it mean for a set of wave functions $\phi_m(x)$, for $m = 1, 2, 3, \ldots$ to be orthonormal? (Give an equation.)

$$\int \phi_m^*(x)\phi_n(x)\,dx = \delta_{mn}$$

(d) What is the general form of the solution to the time independent Schrodinger equation in a region where the potential is constant and greater than the energy, $V_o > E$?

$$\psi(x) = Ae^{\rho x} + Be^{-\rho x},$$

where $\rho = \sqrt{\frac{2m(V_o - E)}{\hbar^2}}$

(e) Write down the continuity equation for the probability.

$$\frac{\partial |\psi|^2}{\partial t} + \frac{\partial j}{\partial x} = 0,$$

where $j = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)$

2. **General properties**
At $t = 0$ the wave function of a particle in an infinite square well between $x = 0$ and $x = a$ is

$$\psi(x,0) = C \text{ for } 0 < x < a/2 \quad (1)$$
$$\psi(x,0) = -C \text{ for } a/2 < x < a. \quad (2)$$

(a) What is the constant C so that the wave function is normalized?

$$1 = \int_0^a |\psi|^2 dx = C^2 \cdot a \rightarrow C = \frac{1}{\sqrt{a}}$$

(b) Compute the expectation values $\langle x \rangle$, $\langle x^2 \rangle$, as well as σ_x for $\psi(x,0)$. You may use the symmetry in the problem.

$$\langle x \rangle = \int_0^a |\psi|^2 x \, dx = \frac{1}{a} \int_0^a x \, dx = \frac{1}{a} \frac{a^2}{2} = \frac{a}{2}$$

$$\langle x^2 \rangle = \int_0^a |\psi|^2 x^2 dx = \frac{1}{a} \int_0^a x^2 dx = \frac{1}{a} \frac{a^3}{3} = \frac{a^2}{3}$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = a \sqrt{\frac{1}{3} - \left(\frac{1}{2}\right)^2} = \frac{a}{\sqrt{12}}$$

(c) Express $\psi(x,0)$ as a linear combination of the eigenstates of the infinite square well, $\psi_n(x)$,

$$\psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x), \tag{3}$$

by calculating the coefficients c_n. Are any of the c_n zero? If so, which ones?

$$C_n = \int_0^{a/2} \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right) \frac{1}{\sqrt{a}}\, dx + \int_{a/2}^{a} \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right) \frac{-1}{\sqrt{a}}\, dx$$

$$= \frac{\sqrt{2}}{a} \left. \frac{-\cos(\pi n x/a)}{\pi n/a} \right|_0^{a/2} + \frac{\sqrt{2}}{a} \left. \frac{\cos(\pi n x/a)}{\pi n/a} \right|_{a/2}^{a}$$

$$= \frac{\sqrt{2}}{\pi n} \Big(\underbrace{\cos(\pi n)}_{2\cos^2\left(\frac{\pi n}{2}\right) - 1} - 2\cos\left(\frac{\pi n}{2}\right) + \underbrace{\cos(0)}_{1} \Big)$$

$$= \frac{\sqrt{2}}{\pi n} \cdot 2\cos\left(\frac{\pi n}{2}\right)\left(\cos\left(\frac{\pi n}{2}\right) - 1\right)$$

This is zero for odd n, and for n a multiple of 4. The non-zero terms are $n = 2, 6, 10, \ldots$ Then $C_n = 4\sqrt{2}/\pi n$.

(d) Use the result of part (c) to write down an expression for $\psi(x,t)$.

$$\psi(x,t) = \sum_{n=2,6,10,\ldots} C_n \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right) e^{-iE_n t/\hbar},$$

where $E_n = \frac{\hbar^2}{2m}\left(\frac{\pi n}{a}\right)^2$

(e) Is there a time, τ, for which $\psi(x,\tau) = -\psi(x,0)$? If so what is that time?

We need $e^{-iE_n \tau/\hbar} = -1$ for $n = 2, 6, 10, \ldots$

$$\rightarrow \frac{\hbar^2}{2m}\left(\frac{\pi \cdot 2}{a}\right)^2 \frac{\tau}{\hbar} = \pi \rightarrow \tau = \frac{2m}{\hbar}\frac{a^2}{4\pi}.$$

3. **Harmonic oscillator**

At $t = 0$ the wave function of a particle in a harmonic oscillator potential is given by

$$\psi(x, 0) = \frac{1}{\sqrt{2}}\psi_1(x) + \frac{e^{i\pi/4}}{\sqrt{2}}\psi_2(x).$$

(a) What is $\psi(x, t)$?

$$\psi(x,t) = \frac{1}{\sqrt{2}}\psi_1(x)\, e^{-i\omega\cdot\frac{3}{2}t} + \frac{e^{i\pi/4}}{\sqrt{2}}\psi_2(x)\, e^{-i\omega\frac{5}{2}t}$$

(b) What is the expectation value of x for $\psi(x, t)$?

$$\langle x\rangle = \int \left(\frac{1}{\sqrt{2}}\psi_1(x)e^{i\omega\frac{3}{2}t} + \frac{e^{-i\pi/4}}{\sqrt{2}}\psi_2(x)e^{i\omega\frac{5}{2}t}\right)$$

$$\sqrt{\frac{\hbar}{2m\omega}}\,(a_- + a_+)\left(\frac{1}{\sqrt{2}}\psi_1(x)e^{-i\omega\frac{3}{2}t} + \frac{e^{i\pi/4}}{\sqrt{2}}\psi_2(x)e^{-i\omega\frac{5}{2}t}\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\,\frac{1}{2}\left(\sqrt{2}\,e^{i\pi/4}e^{-i\omega t} + \sqrt{2}\,e^{-i\pi/4}e^{i\omega t}\right)$$

$$= \sqrt{\frac{\hbar}{m\omega}}\cos\left(\omega t - \frac{\pi}{4}\right)$$

(c) What is the expectation value of p for $\psi(x, t)$?

$\langle p\rangle$ done similarly with $i\sqrt{\frac{\hbar m\omega}{2}}\,(a_+ - a_-)$:

$$\langle p\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\,\frac{1}{2}\left(\sqrt{2}e^{i\omega t}e^{-i\pi/4} - \sqrt{2}e^{-i\omega t}e^{i\pi/4}\right)$$

$$= -\sqrt{\hbar m\omega}\,\sin\left(\omega t - \frac{\pi}{4}\right)$$

(d) What is the expectation value of the energy for $\psi(x,t)$?

$$H = \hbar\omega\left(a_+ a_- + \tfrac{1}{2}\right) \;;\; H\psi_n = \hbar\omega\left(n + \tfrac{1}{2}\right)\psi_n$$

$$\rightarrow \langle E\rangle = \langle H\rangle = \frac{1}{2}\hbar\omega\left(1+\frac{1}{2}\right) + \frac{1}{2}\hbar\omega\left(2+\frac{1}{2}\right) = 2\hbar\omega$$

(e) Determine $\sigma_x\sigma_p$ for $\psi(x,t)$ and show that the uncertainty principle is satisfied.

$$x^2 = \frac{\hbar}{2m\omega}\left(a_-^2 + a_+^2 + a_+a_- + a_-a_+\right)$$

zero for this ψ

$$\rightarrow \langle x^2\rangle = \frac{\hbar}{2m\omega}\langle a_+a_- + a_-a_+\rangle$$

$$= \frac{\hbar}{2m\omega}\left(\frac{1}{2}(2\cdot 1+1) + \frac{1}{2}(2\cdot 2+1)\right) = 2\frac{\hbar}{m\omega}$$

$$\langle p^2\rangle = \frac{\hbar m\omega}{2}\langle a_+a_- + a_-a_+\rangle = 2\hbar m\omega$$

$$\rightarrow \sigma_x = \sqrt{\frac{\hbar}{m\omega}}\sqrt{2-\cos^2\left(\omega t-\frac{\pi}{4}\right)} \geq \sqrt{\frac{\hbar}{m\omega}}$$

$$\sigma_p = \sqrt{\hbar m\omega}\sqrt{2-\sin^2\left(\omega t-\frac{\pi}{4}\right)} \geq \sqrt{\hbar m\omega}$$

$$\rightarrow \sigma_x\sigma_p \geq \hbar > \hbar/2$$

4. **Piecewise constant and delta function potentials**
For this problem use the one dimensional time independent Schrodinger equation with potential

$$V(x) = 0 \text{ for } x < -a \tag{4}$$
$$V(x) = V_o < 0 \text{ for } -a < x < 0 \tag{5}$$
$$V(x) = \infty \text{ for } x > 0. \tag{6}$$

The energy of the particle is in the range $V_o < E < 0$.

(a) For $V_o < E < 0$, what is the general form of the solution? Make sure to define all the variables that you introduce.

$-a < x < 0$: $\psi = Ae^{ikx} + A'e^{-ikx}$, $k = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

$x < -a$: $\psi = Be^{\rho x} + B'e^{-\rho x}$, $\rho = \sqrt{\frac{2m(-E)}{\hbar^2}}$

(b) What is the boundary condition for an infinite square well? This is the same boundary condition as we have in this problem at $x = 0$. Apply this boundary condition to eliminate one of the coefficients in part (a) in the region $-a < x < 0$.

$\psi = 0$ at $x = 0 \rightarrow A + A' = 0$, $A' = -A$

$\rightarrow \psi = A(e^{ikx} - e^{-ikx})$

$= 2iA \sin(kx)$

(c) Which of the solutions in the region $x < -a$ is physical and why?

$Be^{\rho x}$ is physical because it decays as $x \rightarrow -\infty$.

(d) What are the boundary conditions at $x = -a$ applied to the wave function of parts (a)-(c)?

$$2iA\sin(-ka) = Be^{-\rho a}$$

$$k\,2iA\cos(-ka) = \rho Be^{-\rho a}$$

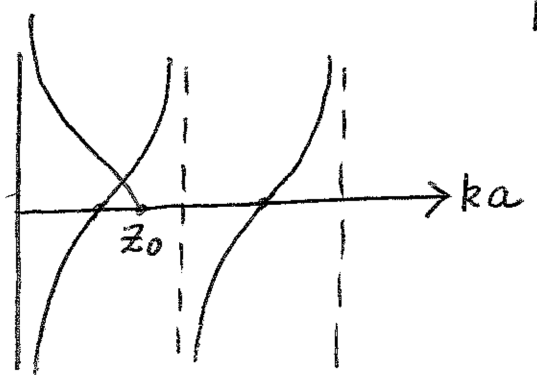
(e) Derive a condition for there to be a bound states solution and solve it graphically. What is the condition that there be one bound state solution?

$$-\cot(ka) = \frac{\rho}{k} = \frac{\rho a}{ka}.$$

$$k^2 + \rho^2 = \frac{2m(E-V_0)}{\hbar^2} + \frac{2m(-E)}{\hbar^2} = \frac{2m(-V_0)}{\hbar^2}$$

Let $z_0^2 = \dfrac{2m(-V_0)a^2}{\hbar^2}$.

Then $-\cot(ka) = \dfrac{\sqrt{z_0^2 - (ka)^2}}{ka}$



There is one bound state solution for $\frac{\pi}{2} \leq z_0 < \frac{3\pi}{2}$