Name:

Exam 2 - PHY 4604 - Fall 2013
November 14, 2013
8:20-10:20PM, NPB 1002

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

$$
\begin{aligned}
Y_0^0 &= \frac{1}{2}\sqrt{\frac{1}{\pi}} \\
Y_1^1(\theta,\varphi) &= \frac{-1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta \\
Y_1^0(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{\pi}}\cdot\cos\theta \\
Y_1^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta \\
Y_2^2(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot e^{2i\varphi}\cdot\sin^2\theta \\
Y_2^1(\theta,\varphi) &= \frac{-1}{2}\sqrt{\frac{15}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta\cdot\cos\theta \\
Y_2^0(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{5}{\pi}}\cdot(3\cos^2\theta-1) \\
Y_2^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{15}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta\cdot\cos\theta \\
Y_2^{-2}(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot e^{-2i\varphi}\cdot\sin^2\theta
\end{aligned}
$$

1. **Short answer section**

(a) What are the momentum eigenvectors, $f_p(x)$? How are they normalized? Specifically, what is $\langle f_p | f_{p'} \rangle$?

$$f_p(x) = \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}$$

$$\langle f_p | f_{p'} \rangle = \delta(p - p')$$

(b) What is the general uncertainty relation for two operators, A and B, which do not commute?

$$\sigma_A \sigma_B \geq \left| \frac{1}{2i} \langle [A, B] \rangle \right|$$

(c) If A, B, and C are Hermitian operators, what is the Hermitian conjugate of $A(B + iC)$?

$$(A(B + iC))^\dagger = (B + iC)^\dagger A^\dagger = (B^\dagger - iC^\dagger)A^\dagger$$
$$= (B - iC)A$$

(d) How is the radial part of the wave function, $R(r)$, related to $u(r)$?

$$R(r) = u(r)/r$$

2. **Formalism**
Consider the operator A below expressed as a matrix.

$$A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

(a) What are the eigenvalues of A?

$$\det \begin{pmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{pmatrix} = (3-\lambda)(-3-\lambda) - 4^2 = 0$$

$$= \lambda^2 - 3^2 - 4^2 = 0$$

$$\rightarrow \lambda \pm 5$$

(b) What are the normalized eigenvectors of A?

$\lambda = 5$: $$\begin{pmatrix} 3-5 & 4 \\ 4 & -3-5 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$$\rightarrow 4c_2 = 2c_1 \rightarrow c_1 = 2c_2$$

$$\rightarrow |\psi_a\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\lambda = -5$: By orthogonality

$$|\psi_b\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

(c) Labeling the eigenvectors as $|\psi_a\rangle$ and $|\psi_b\rangle$, what are the projection operators, $|\psi_a\rangle\langle\psi_a|$ and $|\psi_b\rangle\langle\psi_b|$, for the eigenstates?

$$|\psi_a\rangle\langle\psi_a| = \frac{1}{5}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

$$|\psi_b\rangle\langle\psi_b| = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$$

(d) Can one find simultaneous eigenvectors of A and B, where

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\text{? Why?}$$

$$[A, B] = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix} - \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 6 \\ -6 & 0 \end{pmatrix} \neq 0$$

One can not find simultaneous eigenvalues because $[A,B] \neq 0$.

3. **Measurements**

The Hamiltonian of a three state system is given by

$$H = \hbar\omega \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

It has eigenstates

$$|\psi_a\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{2\pi i/3} \\ 1 \\ e^{-2\pi i/3} \end{pmatrix}, |\psi_b\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, |\psi_c\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{-2\pi i/3} \\ 1 \\ e^{2\pi i/3} \end{pmatrix},$$

and corresponding eigenvalues $E_a = E_c = 2\hbar\omega\cos(2\pi/3) = -\hbar\omega$ and $E_b = 2\hbar\omega$.

(a) If the system is initially in the state

$$|\psi_i\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{1}$$

and an energy measurement is made, what are the possible outcomes of the measurement and probabilities of those outcomes?

outcome	probability
$2\hbar\omega$	$1/3$
$-\hbar\omega$	$2/3$

(b) Suppose instead of making an energy measurement the system is evolved in time for a time t. What is $|\psi(t)\rangle$?

$$|\psi(t)\rangle = \frac{e^{-2\pi i/3}}{\sqrt{3}} e^{+i\omega t} |\psi_a\rangle + \frac{1}{\sqrt{3}} e^{-2i\omega t} |\psi_b\rangle$$

$$+ \frac{e^{2\pi i/3}}{\sqrt{3}} e^{+i\omega t} |\psi_c\rangle = \frac{1}{3}\begin{pmatrix} 2 \\ 2\cos(2\pi/3) \\ 2\cos(-2\pi/3) \end{pmatrix} e^{i\omega t} + \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-2i\omega t}$$

$$= \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} e^{i\omega t} + \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-2i\omega t}$$

(c) At time t what is the probability of being in the state, $|\psi_i\rangle$?

$$\langle\psi_i|\psi(t)\rangle = \frac{e^{-i2\pi i/3}}{\sqrt{3}} e^{i\omega t}\langle\psi_i|\psi_a\rangle \to \frac{1}{3} e^{i\omega t}$$

$$+ \frac{1}{\sqrt{3}} e^{-2i\omega t}\langle\psi_i|\psi_b\rangle \to \frac{1}{3} e^{-2i\omega t}$$

$$+ \frac{e^{2\pi i/3}}{\sqrt{3}} e^{i\omega t}\langle\psi_i|\psi_c\rangle \to \frac{1}{3} e^{i\omega t}$$

$$\text{Probability} = \left|\frac{2}{3} e^{+i\omega t} + \frac{1}{3} e^{-2i\omega t}\right|^2$$

$$= \frac{5}{9} + \frac{4}{9}\cos(3\omega t)$$

(d) Suppose that instead of part (c) energy measurement is made at time t. What are the possible outcomes of an energy measurement and the probabilities of those outcomes?

Same as part (a).

4. **Angular Momentum**

(a) What is the commutator $[L_x L_y, L_z]$ expressed as a product of two operators?

$$[L_x L_y, L_z] = L_x L_y L_z - L_x L_z L_y + L_x L_z L_y - L_z L_x L_y$$

$$= L_x [L_y, L_z] + [L_x, L_z] L_y$$

$$= L_x \, i\hbar L_x \quad - i\hbar L_y L_y$$

$$= i\hbar (L_x^2 - L_y^2)$$

(b) What is the matrix element $\langle 2, -1 | L_x L_y | 2, -1 \rangle$?

$$L_x = \frac{L_+ + L_-}{2} \qquad L_y = \frac{L_+ - L_-}{2i}$$

$$\rightarrow L_x L_y = \frac{1}{4i} (L_+^2 - L_-^2 + \underbrace{[L_-, L_+]}_{-2\hbar L_z})$$

$-2\hbar L_z$, since

$$[L_x - iL_y, L_x + iL_y] = i[L_x, L_y] - i[L_y, L_x].$$

$$\langle 2, -1 | L_x L_y | 2, -1 \rangle = \langle 2, -1 | \frac{-2\hbar L_z}{4i} | 2, -1 \rangle$$

$$= \frac{-2\hbar}{4i} (-\hbar) = -\frac{1}{2} i \hbar^2$$

(c) What is the matrix element $\langle 2,1|L_xL_y|2,-1\rangle$?

From page 7, the non-zero contribution comes from L_+^2

$$\langle 2,1|\frac{L_+^2}{4i}|2,-1\rangle = \frac{1}{4i}\hbar\sqrt{2(2+1)-(-1)(1+-1)} \times\hbar\sqrt{2(2+1)-0(1+0)}$$

$$= \frac{1}{4i}\hbar^2 6 = -\frac{3}{2}i\hbar^2$$

(d) A wavefunction has the form $R(r)x^2$, where $r = \sqrt{x^2+y^2+z^2}$. What are the possible outcomes with *non-zero* probabilities of an L^2 measurement and of an L_z measurement?

In Spherical coordinates $x = r\sin\theta\cos\varphi$ so the angular part of x^2 is $\sin^2\theta\cos^2\varphi$.

$$Y_{2,-2}+Y_{2,2} \propto \sin^2\theta\cos(2\varphi) = \sin^2\theta(2\cos^2\varphi-1).$$

$$= 2\left(\frac{x}{r}\right)^2 - 2\sin^2\theta$$

$$Y_{2,0} \propto 3\cos^2\theta - 1 = 3(1-\sin^2\theta)-1 = \underbrace{2}_{\propto Y_{0,0}} - 3\sin^2\theta$$

Thus, $\frac{x^2}{r^2}$ is a linear combination of

$Y_{2,-2}, Y_{2,2}, Y_{2,0}$, and $Y_{0,0}$.

(non-zero) Outcomes:

L^2	L_z
0	0
$\hbar^2 2(2+1)$	$-2\hbar, 0, 2\hbar$

5. **Radial Schrodinger Equation**
In the following consider the spherically symmetric potential $V(r) = V_o > 0$ for $r_1 < r < r_2$ and $V(r) = \infty$ elsewhere, i.e. for $0 < r < r_1$ and for $r > r_2$.

(a) Solve the radial Schrodinger equation for $u(r)$ in the region $r_1 < r < r_2$ with the above $V(r)$ and $l = 0$. Assume that $E > V_o$.

For $l = 0$, $-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V_o u = E u$

$\rightarrow u(r) = A e^{ikr} + B e^{-ikr}$

with $= k = \sqrt{\frac{2m(E-V_o)}{\hbar^2}}$.

(b) Because the potential is infinity for $r < r_1$ and $r > r_2$ there are boundary conditions for $u(r)$ just as for the one dimensional infinite well. What are the boundary conditions?

$u(r_1) = 0$

$u(r_2) = 0$

(c) Apply the boundary conditions to find a general expression for $u(r)$. Normalize.

$$Ae^{ikr_1} + Be^{-ikr_1} = 0 \rightarrow B = -e^{2ikr_1} A$$

$$\begin{aligned}\rightarrow u(r) &= Ae^{ikr} - Ae^{-ikr}e^{2ikr_1} \\ &= Ae^{ikr_1}\left(e^{ik(r-r_1)} - e^{-ik(r-r_1)}\right) \\ &= 2iAe^{ikr_1}\sin(k(r-r_1))\end{aligned}$$

$$u(r_2) = 0 = 2iAe^{ikr_1}\sin(k(r_2-r_1)) \rightarrow k = \frac{\pi n}{r_2-r_1}$$

Using the normalization condition, $\int_0^\infty u(r)^2 dr = 1$,

$$u(r) = \sqrt{\frac{2}{r_2-r_1}}\sin(k(r-r_1)) \text{ with } k = \frac{\pi n}{r_2-r_1}.$$

(d) What are the energies of the eigenstates?

$$E_n = \frac{\hbar^2 k^2}{2m} + V_0 \quad \text{with } k = \frac{\pi n}{r_2-r_1}$$