Name:

Final Exam - PHY 4604 - Fall 2013

December 12, 2013

10:00A-12:00P, BRY 130

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

Each exam question, (a),..., (d), is worth 5 points, and the entire exam is out of 100 points. Some formulas are given with the relevant question.

1. **Short answer:**

(a) Write down the time dependent Schrodinger equation in one dimension.

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi$$

(b) What are the energies and wave functions of an infinite square well in one dimension with width a?

$\psi(x) = \sqrt{\frac{2}{a}} \sin(kx)$,
where $k = \frac{\pi n}{a}$ and
$E = \frac{\hbar^2 k^2}{2m}$.

(c) How many Hydrogen atom states have $n = 3$ and $l = 1$? Include the spin degeneracy.

For $l = 1$, $m = -1, 0, 1$.
Including the spin degeneracy,
$3 \times 2 = 6$ states.

(d) Define a complete set of states.

ψ_n form a complete set of states if any state, ψ, can be expressed as a linear combination of the ψ_n:

$$\psi = \sum_n c_n \psi_n.$$

2. **One Dimensional Schrodinger Equation:**
Consider the potential

$$\begin{aligned} V(x) &= V_1 \text{ for } x < 0 \\ V(x) &= V_2 \text{ for } x > 0, \end{aligned}$$

where $V_1 < V_2$. In the following take the energy to be larger than the potentials, $E > V_2 > V_1$.

(a) What is the general form of the solutions for $x < 0$ and $x > 0$? Make sure to specify any wave vectors in terms of V_1 and V_2.

$x<0: \ \psi(x) = Ae^{ik_1 x} + Be^{-ik_1 x}, \ k_1 = \sqrt{\frac{2m(E-V_1)}{\hbar^2}}$

$x>0: \ \psi(x) = Ce^{ik_2 x} + De^{-ik_2 x}, \ k_2 = \sqrt{\frac{2m(E-V_2)}{\hbar^2}}$

(b) What is the form of a solution corresponding to a wave going incident from the right ($x > 0$) and partially transmitted to the left ($x < 0$)?

$x<0: \ \psi(x) = Be^{-ik_1 x}$

$x>0: \ \psi(x) = Ce^{ik_2 x} + De^{-ik_2 x}$

(c) Match boundary conditions at $x = 0$ and solve for the wave function of part (b) up to an overall multiplicative factor.

Take $D = 1$.

$$\psi(0) = B = C + 1 \quad \rightarrow B - C = 1$$

$$\psi'(0) = -ik_1 B = ik_2 C - ik_2 \rightarrow B + \frac{k_2}{k_1} C = \frac{k_2}{k_1}$$

$$\rightarrow \left(\frac{k_2}{k_1} + 1\right) C = \left(\frac{k_2}{k_1} - 1\right) \rightarrow C = \frac{\frac{k_2}{k_1} - 1}{\frac{k_2}{k_1} + 1} = \frac{k_2 - k_1}{k_2 + k_1}$$

$$B = 1 + C = \frac{2k_2}{k_2 + k_1}$$

(d) What are the transmission probability and reflection probabilities?

With $D = 1$,

$$T = \frac{k_1}{k_2} |B|^2 = \frac{4k_1 k_2}{(k_1 + k_2)^2}$$

$$R = |C|^2 = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}.$$

Note that $R + T = 1$.

3. **Harmonic Oscillator:**
The Hamiltonian of the one-dimensional harmonic oscillator is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2.$$

Suppose state of system at $t = 0$ is

$$\psi(x, 0) = c_n \psi_n(x) + e^{i\varphi} c_{n+1} \psi_{n+1}(x),$$

where $\psi_n(x)$ for $n = 0, 1, 2, \ldots$ are the energy eigenstates of the Hamiltonian. The coefficients c_n and c_{n+1}, are real and the wave function is normalized, $c_n^2 + c_{n+1}^2 = 1$.

(a) What is the wave function, $\psi(x, t)$, at time $t > 0$?

$$\psi(x,t) = C_n e^{-i(n+\frac{1}{2})\omega t} \psi_n(x) + C_{n+1} e^{-i(n+\frac{3}{2})\omega t} e^{i\varphi} \psi_{n+1}(x)$$

(b) Compute the expectation values of x at time t, where

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-).$$

What is the amplitude of the position oscillation?

$$\frac{\langle x\rangle}{\sqrt{\hbar/2m\omega}} = C_n e^{i(n+\frac{1}{2})\omega t} C_{n+1} e^{-i(n+\frac{3}{2})\omega t} e^{i\varphi} \langle n|\overbrace{a_-}^{\sqrt{n+1}}|n+1\rangle + C_{n+1} e^{i(n+\frac{3}{2})\omega t} e^{-i\varphi} C_n e^{-i(n+\frac{1}{2})\omega t} \langle n+1|\underbrace{a_+}_{\sqrt{n+1}}|n\rangle$$

$$= C_n C_{n+1}\sqrt{n+1}\left(e^{-i\omega t} e^{i\varphi} + e^{i\omega t} e^{-i\varphi}\right)$$

$$\rightarrow \langle x\rangle = 2\sqrt{\frac{\hbar}{2m\omega}}\, C_n C_{n+1}\sqrt{n+1}\, \cos(\omega t - \varphi)$$

The amplitude is $2\sqrt{\frac{\hbar}{2m\omega}}\, C_n C_{n+1}\sqrt{n+1}$.

(c) Compute the expectation values of p at time t, where

$$p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-).$$

What is the amplitude of the momentum oscillation?

$$\langle p \rangle = i\sqrt{\frac{\hbar m\omega}{2}}\, c_{n+1} e^{i(n+\frac{3}{2})\omega t} e^{-i\varphi} c_n e^{i(n+\frac{1}{2})\omega t} \sqrt{n+1}$$

$$- i\sqrt{\frac{\hbar m\omega}{2}}\, c_n e^{i(n+\frac{1}{2})\omega t} c_{n+1} e^{-i(n+\frac{3}{2})\omega t} e^{i\varphi} \sqrt{n+1}$$

$$= -2\sqrt{\frac{\hbar m\omega}{2}}\, c_n c_{n+1} \sqrt{n+1}\, \sin(\omega t - \varphi)$$

The amplitude is $2\sqrt{\frac{\hbar m\omega}{2}}\, c_n c_{n+1} \sqrt{n+1}$.

(d) Compute the expectation values of the energy at time t. How is the expectation value of the energy related to the amplitudes of the position and momentum oscillations? Is this the same as the classical relation or different?

$$\langle E \rangle = \hbar\omega(n+\tfrac{1}{2})c_n^2 + \hbar\omega(n+\tfrac{3}{2})c_{n+1}^2$$

Classically, the energy is proportional to the amplitudes squared. In quantum mechanics that is not the case. Even for zero amplitude, e.g. $c_{n+1} = 0$, the energy is not zero.

4. **Formalism:**
The Pauli spin matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) At $t = 0$ a measurement of $S_x = (\hbar/2)\sigma_x$ yields $-\hbar/2$. What is the state of the system?

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ since } \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

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outcome	probability
$\hbar/2$	$\frac{1}{2}\frac{1}{1+(\sqrt{2}-1)^2}\left\lvert (1 \quad \sqrt{2}-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\rvert^2 = \frac{(2-\sqrt{2})^2}{2(4-2\sqrt{2})} = \frac{3-2\sqrt{2}}{4-2\sqrt{2}}$
$-\hbar/2$	$\frac{1}{2}\frac{1}{1+(\sqrt{2}-1)^2}\left\lvert (\sqrt{2}-1 \quad -1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\rvert^2 = \frac{2}{2(4-2\sqrt{2})} = \frac{1}{4-2\sqrt{2}}$

$\frac{1}{4+2\sqrt{2}}$

$\frac{3+2\sqrt{2}}{4+2\sqrt{2}}$

(b) Right after the S_x measurement is made, a second measurement is made of $(S_x + S_z)/\sqrt{2}$. What are the possible outcomes of those measurements and their probabilities?

$$\frac{1}{\sqrt{2}}(S_x + S_z) = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$$\det\begin{pmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} = -(1^2 - \lambda^2) - 1 \xrightarrow{=0} \lambda^2 - 2 = 0 \rightarrow \lambda = \pm\sqrt{2}$$

$$\begin{pmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \propto \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$$

$\rightarrow$ The eigenvalues and eigenvectors of $\frac{1}{\sqrt{2}}(S_x + S_z)$ are

$$+\frac{\hbar}{2}, \quad \frac{1}{\sqrt{1^2+(\sqrt{2}-1)^2}}\begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$$

$$-\frac{\hbar}{2}, \quad \frac{1}{\sqrt{1+(\sqrt{2}-1)^2}}\begin{pmatrix} \sqrt{2}-1 \\ -1 \end{pmatrix}$$

(c) Suppose that the result of the measurement in part (b) is positive. What is the state of the system then?

$$\frac{1}{\sqrt{\underbrace{1+(\sqrt{2}-1)^2}_{4-2\sqrt{2}}}}\begin{pmatrix}1\\ \sqrt{2}-1\end{pmatrix} = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix}1+\sqrt{2}\\ 1\end{pmatrix}$$

(d) If a final measurement is made of $S_{\cancel{y}}$ S_x now, what is the probability that one is in the same state as in part (a)?

$$\left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}\frac{1}{\sqrt{1+(\sqrt{2}-1)^2}}\begin{pmatrix}1\\ \sqrt{2}-1\end{pmatrix}\right|^2 = \frac{3-2\sqrt{2}}{4-2\sqrt{2}} = \frac{1}{4+2\sqrt{2}}$$

5. **Angular momentum:**

(a) Compute the commutator $[(L_z)^2, (L_x)^2]$?

$$= L_z^2 L_x L_x - L_x L_x L_z^2$$
$$- L_x L_z^2 L_x + L_x L_z^2 L_x$$
$$= [L_z^2, L_x] L_x + L_x [L_z^2, L_x].$$

$$[L_z^2, L_x] = L_z L_z L_x - L_z L_x L_z + L_z L_x L_z - L_x L_z L_z$$
$$= L_z [L_z, L_x] + [L_z, L_x] L_z$$
$$= i\hbar L_z L_y + i\hbar L_y L_z$$

$$\rightarrow [L_z^2, L_x^2] = i\hbar (L_z L_y L_x + L_y L_z L_x + L_x L_z L_y + L_x L_y L_z)$$

(b) What is $\langle 1, 1|(J_x)^2 - (J_y)^2|1, -1\rangle$?

$$\left.\begin{array}{l} J_+ = J_x + i J_y \\ J_- = J_x - i J_y \end{array}\right\} \rightarrow \begin{array}{l} J_x = \frac{1}{2}(J_+ + J_-) \\ J_y = \frac{1}{2i}(J_+ - J_-) \end{array}$$

$$J_x^2 - J_y^2 = \frac{1}{4}(J_+ + J_-)^2 + \frac{1}{4}(J_+ - J_-)^2$$
$$= \frac{1}{2}(J_+^2 + J_-^2)$$

$$J_+ |1, -1\rangle = \hbar\sqrt{1(1+1) + 1(1-1)}\, |1, 0\rangle = \hbar\sqrt{2}\, |1, 0\rangle$$
$$J_+ |1, 0\rangle = \hbar\sqrt{2}\, |1, 1\rangle$$

$$\rightarrow \langle 1, 1| J_x^2 - J_y^2 | 1, -1\rangle = \frac{\hbar^2 2}{2} = \hbar^2.$$

(c) What is $\langle 1, 1|(J_x)^2 - (J_y)^2|1, 1\rangle$?

0 since $J_x^2 - J_y^2 = \frac{1}{2}(J_+^2 + J_-^2)$

(d) What is the state with $j = 4$ and $m = 3$ created by adding the angular momentum of two spin 2 particles? Hint: The state with $j = 4$ and $m = 4$ is $|2, 2\rangle|2, 2\rangle$.

$$J_-|4,4\rangle = \hbar\sqrt{4(4+1) - 4(4-1)}\,|4,3\rangle = \hbar\sqrt{8}\,|4,3\rangle$$

$$J_-^{(1)}|2,2\rangle = \hbar\sqrt{2(2+1) - 2(2-1)}\,|2,1\rangle = \hbar\, 2\,|2,1\rangle$$

$$\rightarrow |4,4\rangle = \frac{1}{\sqrt{2}}\left(|2,2\rangle|2,1\rangle + |2,1\rangle|2,2\rangle\right)$$