Free particle: $(V=0)$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+\frac{\hbar^2 \ell(\ell+1)}{2mr^2}u = Eu \quad , \boxed{R(r)=u(r)/r}$$

$$\frac{d^2u}{dr^2}-\frac{\ell(\ell+1)}{r^2}u+\frac{2mE}{\hbar^2}u=0$$

Let $k=\sqrt{\frac{2mE}{\hbar^2}}$ $\quad (E>0)$

Then $\frac{d^2u}{dr^2}-\frac{\ell(\ell+1)}{r^2}u+k^2u=0.$

Let $\rho = kr$

$$\rightarrow \boxed{\frac{d^2u}{d\rho^2}-\frac{\ell(\ell+1)}{\rho^2}u+u=0}$$

and

$$\boxed{\frac{1}{\rho}d^2(\rho R)-\frac{\ell(\ell+1)}{\rho^2}R+R=0}$$

For $\ell = 0$ we know the solution:

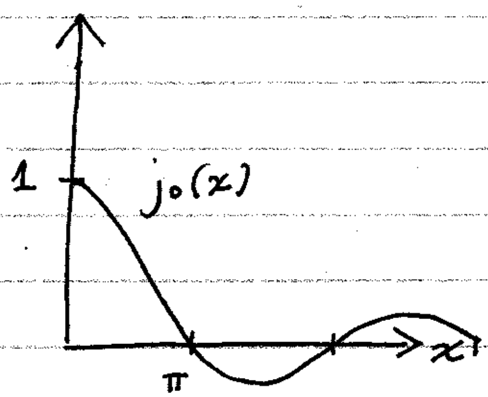
$$\frac{d^2u}{d\rho^2} + u = 0 \rightarrow u(\rho) = \sin(\rho), \cos(\rho)$$

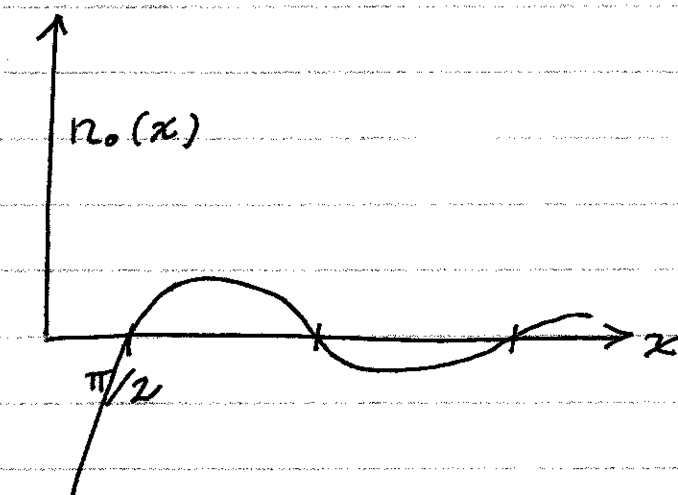
$$R(\rho) = \frac{\sin(\rho)}{\rho}, \frac{\cos(\rho)}{\rho}$$

For $\ell \neq 0$ the solutions for R are spherical Bessel functions: (Table 4.4)

$\ell = 0$ $\quad j_0(x) = \dfrac{\sin x}{x} \qquad n_0(x) = -\dfrac{\cos(x)}{x}$

$\ell = 1$ $\quad j_1(x) = \dfrac{\sin x}{x^2} - \dfrac{\cos x}{x} \qquad n_1(x) = -\dfrac{\cos(x)}{x^2} - \dfrac{\sin x}{x}$





The $j_\ell(x)$ will be non-singular as $x \to 0$:

$$j_\ell(x) \sim x^\ell \quad \text{as } x \to 0$$

$$n_\ell(x) \sim \frac{1}{x^{\ell+1}} \quad \text{as } x \to 0$$

Infinite spherical well:

$$V(r) = 0 \qquad r < a$$
$$V(r) = \infty \qquad r > a$$

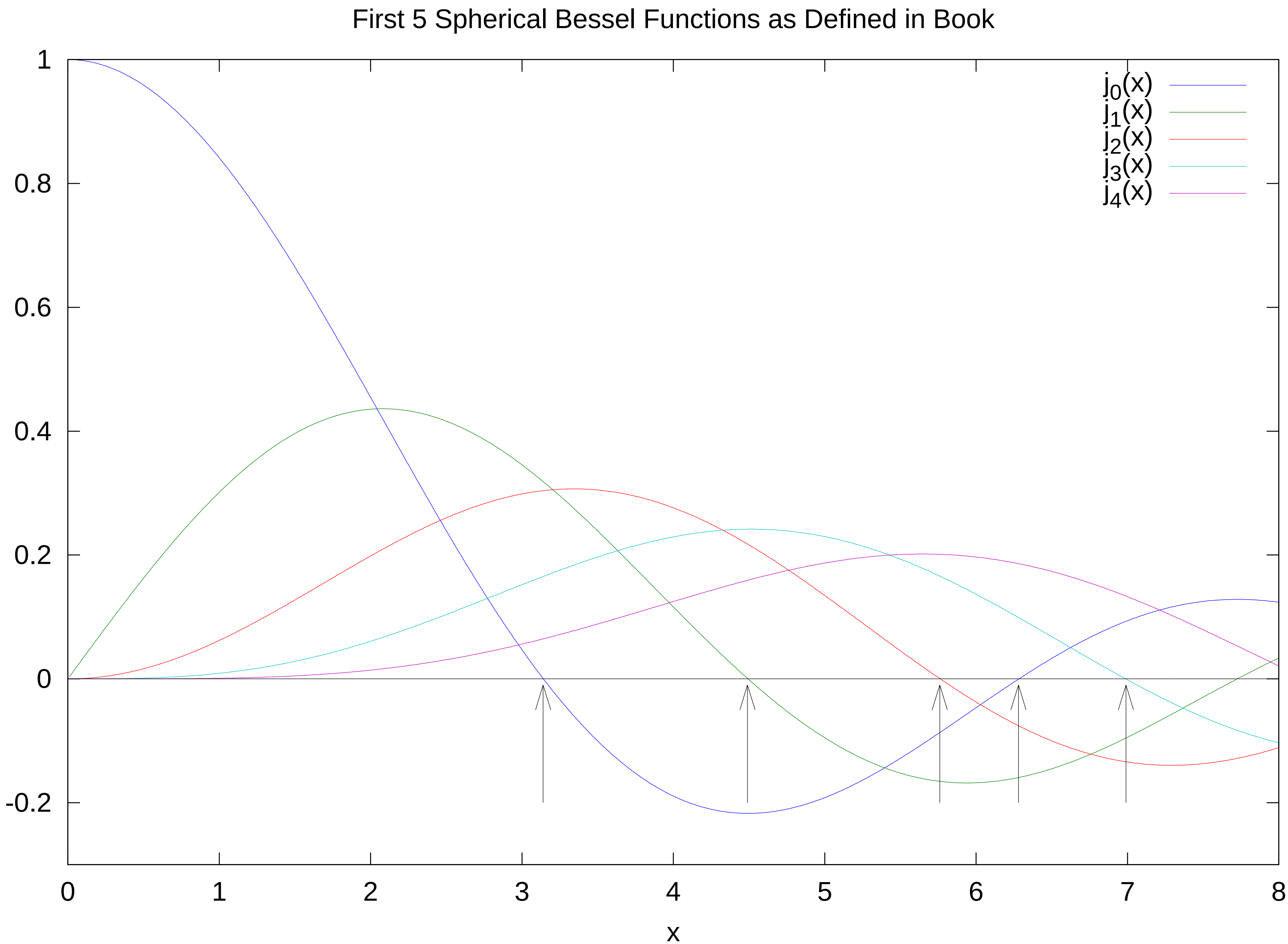
$R(a) = 0$, $R(r<a) \propto j_\ell(ka) \rightarrow j_\ell(ka) = 0$

with $E = \frac{\hbar^2 k^2}{2m}$

The zeros of the first few j_ℓ are plotted on the following page:

			degeneracy
①	first zero: $j_0(ka) = 0$	, $ka \approx 3.14$ ← π	$2 \times 1 = 2$
②	2nd zero: $j_1(ka) = 0$	, $ka \approx 4.49$	$2 \times 3 = 6$
③	3rd zero: $j_2(ka) = 0$	, $ka \approx 5.76$	$2 \times 5 = 10$
④	4th zero: $j_0(ka) = 0$	, $ka \approx 6.28$ ← 2π	$2 \times 1 = 2$
⑤	5th zero: $j_3(ka) = 0$	, $ka \approx 6.99$	$2 \times 7 = 14$

↑ spin


First 5 Spherical Bessel Functions as Defined in Book
1
0.8
0.6
0.4
0.2
0
-0.2
0
1
2
3
4
5
6
7
8
x
$j_0(x)$
$j_1(x)$
$j_2(x)$
$j_3(x)$
$j_4(x)$

In the nuclear shell model one imagines filling a well like this one with nucleons (protons & neutrons).

When a shell is full, the nucleus is particularly stable because it requires extra energy to add another nucleon. This is analagous to why the noble gases (He, Ne, Ar, ...) are not chemically reactive.

The number of nucleons when a filled shell occurs are the so-called magic numbers.

Shells filled	Infinite 3D well	Observed (by stability, more tightly bound)
1	$2 = 2$	2
1,2	$2+6 = 8$	8
1,2,3	$2+6+10 = 18$	20
1,2,3,4	$2+6+10+2 = 20$	28

With refinements to the potential & including some interactions one can obtain the observer magic numbers.