Two examples:

In addition to finishing the previous lecture notes we did two examples:

For $\psi_0$:

$$\langle x^2\rangle = \int \psi_0^*(x) \frac{\hbar}{2m\omega} (a_+ + a_-)(a_+ + a_-) \psi_0(x)\, dx$$

$$= \int \psi_0^*(x) \frac{\hbar}{2m\omega} (a_+^2 + a_-a_+ + a_+a_- + a_-^2) \psi_0\, dx$$

Since $a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$ & $a_-\psi_n = \sqrt{n}\,\psi_{n-1}$,

$a_+^2\psi_0 = \sqrt{2}\,\psi_2$, which is orthogonal to $\psi_0$

$a_-^2\psi_0 = 0$

$a_+a_-\psi_0 = 0$

$a_-a_+\psi_0 = (\sqrt{1})^2\psi_0$ ... only non-zero term

$$\rightarrow \langle x^2\rangle = \frac{\hbar}{2m\omega}$$

$$\langle p^2\rangle = \int \psi_0^*(x) -\frac{\hbar m\omega}{2} (a_+ - a_-)(a_+ - a_-) \psi_0\, dx$$

$$= \int \psi_0^*(x) \frac{\hbar\omega m}{2} (-a_+^2 + a_+a_- + a_-a_+ - a_-^2) \psi_0\, dx$$

$$= \frac{\hbar\omega m}{2}$$

Note that $\langle H\rangle = \frac{\langle p^2\rangle}{2m} + \frac{1}{2}m\omega^2\langle x^2\rangle = \frac{\hbar\omega}{2}$.