Magnetic Resonance:

Suppose the magnetic field depends on time:

$$H = -\vec{M}\cdot\vec{B} = -\gamma\vec{S}\cdot\vec{B}(t).$$

$$i\hbar\frac{d\langle\vec{S}\rangle}{dt} = i\hbar\frac{d}{dt}\langle\psi(t)|\vec{S}|\psi(t)\rangle = \langle\psi(t)|\vec{S}\,H(t) - H(t)\,\vec{S}\,|\psi(t)\rangle$$
$$= \langle\psi(t)|[\vec{S}, H(t)]|\psi(t)\rangle$$
$$= -\gamma\,\langle\psi(t)|[\vec{S}, \vec{S}\cdot\vec{B}]|\psi(t)\rangle$$

Since $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ and $[\sigma_\alpha, \sigma_\beta] = 2i\,\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$,

$$\rightarrow [S_\alpha, S_\beta] = i\hbar\,\varepsilon_{\alpha\beta\gamma}\vec{S}_\gamma$$

$$\rightarrow i\hbar\frac{dS_\alpha}{dt} = -\gamma\, i\hbar\,\varepsilon_{\alpha\beta\rho}B_\beta S_\rho$$

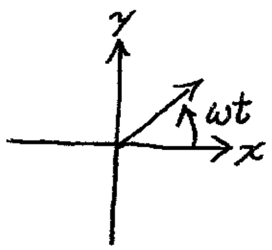
$$\rightarrow \frac{dS_\alpha}{dt} = -\gamma\,\varepsilon_{\alpha\beta\rho}B_\beta S_\rho$$

$$\text{or. } \frac{d\vec{S}}{dt} = -\gamma\,\vec{B}\times\vec{S} = \gamma\,\vec{S}\times\vec{B}$$

Let $\vec{B} = B_0\hat{z}$ for starters.

$$\frac{d\langle S_x\rangle}{dt} = +\gamma B_0\langle S_y\rangle$$
$$\frac{d\langle S_y\rangle}{dt} = -\gamma B_0\langle S_x\rangle$$
$$\frac{d\langle S_z\rangle}{dt} = 0$$

This has solution:



$$\langle S_x\rangle(t) = \cos(\omega_0 t)\langle S_x\rangle(0) - \sin(\omega_0 t)\langle S_y\rangle(0)$$
$$\langle S_y\rangle(t) = \cos(\omega_0 t)\langle S_y\rangle(0) + \sin(\omega_0 t)\langle S_x\rangle(0)$$
$$\langle S_z\rangle(t) = \langle S_z\rangle(0),$$

where $\omega_0 = -\gamma B_0$.

Next consider $\vec{B} = B_0 \hat{z} + B_1(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y})$.

$$\langle \vec{S} \rangle \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \langle S_x \rangle & \langle S_y \rangle & \langle S_z \rangle \\ B_1 \cos(\omega t) & B_1 \sin(\omega t) & B_0 \end{vmatrix}$$

$$= \hat{x}\left(\langle S_y \rangle B_0 - \langle S_z \rangle B_1 \sin(\omega t)\right)$$
$$+ \hat{y}\left(\langle S_z \rangle B_1 \cos(\omega t) - \langle S_x \rangle B_0\right)$$
$$+ \hat{z}\left(\langle S_x \rangle B_1 \sin(\omega t) - \langle S_y \rangle B_1 \cos(\omega t)\right)$$

$$\rightarrow \frac{d\langle S_x \rangle}{dt} = \underbrace{\gamma B_0}_{-\omega_0} \langle S_y \rangle - \underbrace{\gamma B_1}_{-\omega_1} \sin(\omega t) \langle S_z \rangle$$

$$\frac{d\langle S_y \rangle}{dt} = -\gamma B_0 \langle S_x \rangle + \gamma B_1 \cos(\omega t) \langle S_z \rangle$$

$$\frac{d\langle S_z \rangle}{dt} = \gamma B_1 \sin(\omega t) \langle S_x \rangle - \gamma B_1 \cos(\omega t) \langle S_y \rangle$$

To solve these equations switch to a reference frame rotating at angular frequency ω:

$$S_x' \equiv \cos(\omega t)\langle S_x\rangle + \sin(\omega t)\langle S_y\rangle$$

$$S_y' \equiv \cos(\omega t)\langle S_y\rangle - \sin(\omega t)\langle S_x\rangle$$

$$S_z' \equiv \langle S_z\rangle .$$

$$\frac{dS_x'}{dt} = -\omega\sin(\omega t)\langle S_x\rangle + \omega\cos(\omega t)\langle S_y\rangle$$
$$+\cos(\omega t)\{-\omega_0\langle S_y\rangle + \omega_1\sin(\omega t)\langle S_z\rangle\}$$
$$+\sin(\omega t)\{+\omega_0\langle S_x\rangle - \omega_1\cos(\omega t)\langle S_z\rangle\}$$

$$= -(\omega-\omega_0)\sin(\omega t)\langle S_x\rangle + (\omega-\omega_0)\cos(\omega t)\langle S_y\rangle$$
$$= +(\omega-\omega_0)\, S_y'$$

$$\frac{dS_y'}{dt} = -\omega\sin(\omega t)\langle S_y\rangle - \omega\cos(\omega t)\langle S_x\rangle$$
$$+\cos(\omega t)\{+\omega_0\langle S_x\rangle - \omega_1\cos(\omega t)\langle S_z\rangle\}$$
$$-\sin(\omega t)\{-\omega_0\langle S_y\rangle + \omega_1\sin(\omega t)\langle S_z\rangle\}$$
$$= -(\omega-\omega_0)\sin(\omega t)\langle S_y\rangle - (\omega-\omega_0)\cos(\omega t)\langle S_x\rangle - \omega_1\langle S_z\rangle$$
$$= -(\omega-\omega_0)\, S_x' - \omega_1\, S_z'$$

$$\frac{dS_z'}{dt} = +\omega_1\, S_y'$$

Let $\Delta\omega = \omega - \omega_0$. Then

$$\frac{dS_x'}{dt} = \Delta\omega\, S_y'$$

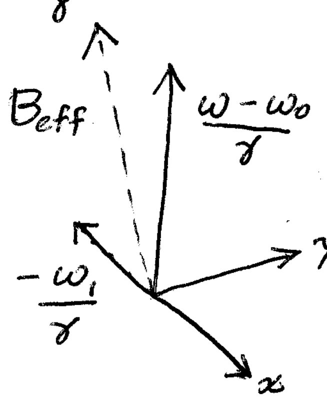
$$\frac{dS_y'}{dt} = -\Delta\omega S_x' - \omega_1 S_z'$$

$$\frac{dS_z'}{dt} = \omega_1 S_y' \quad .$$

$$\rightarrow \frac{d\vec{S}'}{dt} = -\underbrace{(\Delta\omega\hat{z} - \omega_1\hat{x})}_{\gamma\vec{B}_{eff}} \times \vec{S}' = -\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\omega_1 & 0 & \Delta\omega \\ S_x' & S_y' & S_z' \end{vmatrix}$$

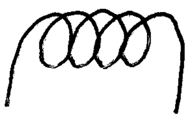
There is an effective field in the rotating frame of

$$B_{eff} = \frac{1}{\gamma}\{\Delta\omega\hat{z} - \omega_1\hat{x}\}.$$



The spin precesses about Beff in the rotating frame.

NMR = nuclear magnetic resonance



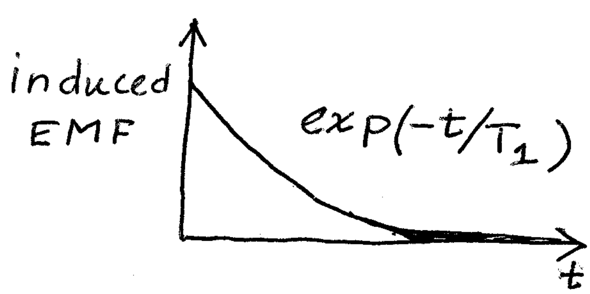
Radio frequency coil produces

$$B_1 \hat{x} \cos(\omega t) = \frac{1}{2}\{ B_1 \cos(\omega t)\hat{x} + B_1 \sin(\omega t)\hat{y} \} + \frac{1}{2}\{ B_1 \cos(\omega t)\hat{x} - B_1 \sin(\omega t)\hat{y} \}.$$

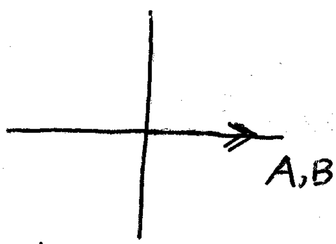
For $\omega \approx \omega_0$ apply a $\frac{\pi}{2}$ pulse, $\omega_1 t = \frac{\pi}{2} \rightarrow t = \frac{\pi}{2\omega_1}$.

$\uparrow S'_z$ only goes to $\longrightarrow S'_y$ only.

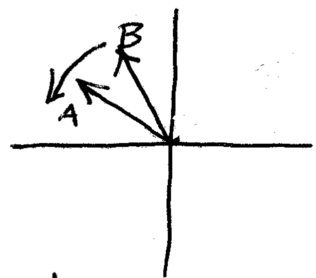
A precessing (moment) spin in the x-y plane produces an induced EMF in the coil, which can be detected. As the spins relax back to the $\hat{z}$ direction the signal decays.



If B_0 is slightly non-uniform, then the signal will also decay because the spins in the x-y plane become dephased:

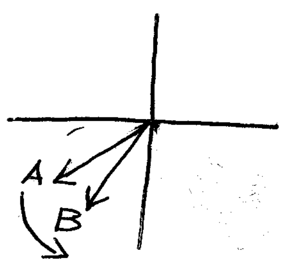


initially

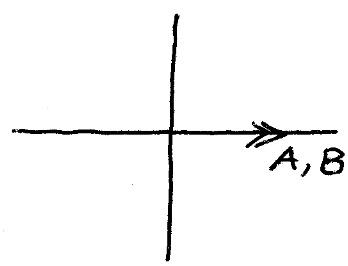


later
(after many revolutions)

The time scale for this is called T_2. It can be reversed by a π pulse.



after π pulse
(note B is ahead)



Spin echo