Name:

Exam 1 - PHY 4604 - Fall 2017

October 4, 2017

8:20-10:10PM, NPB 1002

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

Unless otherwise noted all parts (a), (b), ... are worth 5 points, and the entire exam is 100 points.

Harmonic oscillator:

$$
\begin{aligned}
a_+ &= \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x) \\
a_- &= \frac{1}{\sqrt{2\hbar m\omega}}(+ip + m\omega x) \\
x &= \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \\
p &= i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-) \\
\psi_0(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\
\psi_n &= \frac{1}{\sqrt{n!}}\,(a_+)^n\,\psi_0
\end{aligned}
$$

1. **Short answer section**

(a) Write down the time dependent Schrodinger equation in one dimension.

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\Psi(x,t)$$

(b) What are the normalized solutions to the time independent Schrodinger equation for an infinite square between $0 \leq x \leq a$?

$$\Psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \text{ for } n = 1, 2, 3, \ldots$$

(c) Give an expression for the probability current.

$$j = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)$$

(d) What is the criterion that a set of states $\psi_n(x)$ are complete on $-\infty < x < \infty$?

Any $f(x)$ can be written as

$$f(x) = \sum_n c_n \Psi_n(x).$$

(e) What are the commutators, $[a_-, a_+]$ and $[a_+, a_-]$?

$[a_-, a_+] = 1$

$[a_+, a_-] = -1$

2. **General properties**
For the following it will be helpful to know the integral

$$\int_0^\infty u^n e^{-u} du = n!$$

Consider the wave function $\psi(x) = C\,x\,\exp(-\frac{x}{2a})$ **on the interval** $0 \le x < \infty$.

(a) What is the constant C so that the wave function is normalized?

$$1 = C^2 \int_0^\infty dx\, x^2 e^{-x/a} = C^2 a^3 \int_0^\infty du\, u^2 e^{-u}$$

$$= 2C^2 a^3 \rightarrow \boxed{C = \frac{1}{\sqrt{2}} \frac{1}{a^{3/2}}}$$

$$C^2 = \frac{1}{2a^3}$$

(b) Compute the expectation values $\langle x \rangle$, $\langle x^2 \rangle$, as well as σ_x for $\psi(x, 0)$.

$$\langle x \rangle = C^2 \int_0^\infty dx\, x^3 e^{-x/a} = \frac{1}{2a^3} a^4 \int_0^\infty du\, u^3 e^{-u} = \frac{3!}{2} a = \boxed{3a}$$

$$\langle x^2 \rangle = C^2 \int_0^\infty dx\, x^4 e^{-x/a} = \frac{1}{2a^3} a^5 \int_0^\infty du\, u^4 e^{-u} = \frac{4!}{2} a^2 = \boxed{12a^2}$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{12a^2 - 9a^2} = \boxed{\sqrt{3} \cdot a}$$

(c) Compute the expectation values $\langle p \rangle$, $\langle p^2 \rangle$, as well as σ_p for $\psi(x, 0)$.

$$\frac{d}{dx} x e^{-x/2a} = e^{-x/2a} - \frac{x}{2a} e^{-x/2a} = \left(1 - \frac{x}{2a}\right) e^{-x/2a}$$

$$\frac{d}{dx} \left(1 - \frac{x}{2a}\right) e^{-x/2a} = -\frac{1}{2a}\left(1 - \frac{x}{2a}\right) e^{-x/2a} - \frac{1}{2a} e^{-x/2a}$$

$$= -\frac{1}{a}\left(1 - \frac{x}{4a}\right) e^{-x/2a}$$

(as expected)

$$\rightarrow \langle p \rangle = \frac{\hbar}{i} \frac{1}{2a^3} a^2 \int_0^\infty du \left(u - \frac{u^2}{2}\right) e^{-u} = \frac{\hbar}{i} \frac{1}{2a} \left(1! - \frac{2!}{2}\right) = \boxed{0}$$

$$\langle p^2 \rangle = \hbar^2 \frac{1}{2a^3} \frac{a^2}{a} \int_0^\infty du \left(u - \frac{u^2}{4}\right) e^{-u} = \frac{\hbar^2}{2a^2} \left(1! - \frac{2!}{4}\right) = \boxed{\frac{\hbar^2}{4a^2}}$$

$$\sigma_p = \sqrt{\frac{\hbar^2}{4a^2}} = \boxed{\frac{\hbar}{2a}}$$

(d) Use the results of (b) and (c) above to check that the uncertainty principle is satisfied.

$$\sigma_x \sigma_p = \sqrt{3}\, a \cdot \frac{\hbar}{2a} = \frac{\sqrt{3}}{2} \hbar > \frac{\hbar}{2}$$

(e) What is the commutator, $[p, x^3]$?

Let f be any function.

$$[p, x^3] f = \frac{\hbar}{i} \frac{d}{dx} (x^3 f) - x^3 \frac{\hbar}{i} \frac{df}{dx}$$

$$= \frac{\hbar}{i} 3x^2 f + \frac{\hbar}{i} x^3 \frac{df}{dx} - x^3 \frac{\hbar}{i} \frac{df}{dx} = \frac{\hbar}{i} 3x^2 f$$

$$\rightarrow \boxed{[p, x^3] = \frac{\hbar}{i} 3x^2}$$

3. **Harmonic oscillator**

At $t = 0$ the wave function of a particle in a harmonic oscillator potential is given by

$$\psi(x,0) = \frac{1}{\sqrt{2}}(\psi_n(x) - i\,\psi_{n+1}(x)).$$

(a) What is $\psi(x,t)$?

$$\psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_n(x) e^{-i(n+\frac{1}{2})\omega t} + -i\,\psi_{n+1}(x) e^{-i(n+\frac{3}{2})\omega t}\right)$$

(b) What is the expectation value of the momentum for $\psi(x,t)$? What is the amplitude of the oscillations in $\langle p \rangle$?

$$\langle p \rangle = \frac{1}{2} i \sqrt{\frac{\hbar m \omega}{2}} \int dx \left(\psi_n e^{i(n+\frac{1}{2})\omega t} + i\,\psi_{n+1} e^{i(n+\frac{3}{2})\omega t}\right)$$

$$(a_+ - a_-)$$

$$\left(\psi_n e^{-i(n+\frac{1}{2})\omega t} - i\,\psi_{n+1} e^{-i(n+\frac{3}{2})\omega t}\right)$$

$$= \frac{1}{2} i \sqrt{\frac{\hbar m \omega}{2}}\, i \left(\sqrt{n+1}\, e^{i\omega t} + \sqrt{n+1}\, e^{-i\omega t}\right)$$

$$\langle p \rangle = \underbrace{-\sqrt{\frac{\hbar m \omega}{2}} \sqrt{n+1}}_{\text{Amplitude}} \cos(\omega t)$$

(c) What is the expectation value of the position for $\psi(x,t)$? What is the amplitude of the oscillations in $\langle x \rangle$?

$$\langle x \rangle = \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}} \int dx \left(\psi_n e^{i(n+\frac{1}{2})\omega t} + i\psi_{n+1} e^{i(n+\frac{3}{2})\omega t}\right)$$

$$(a_+ + a_-)$$

$$\left(\psi_n e^{-i(n+\frac{1}{2})\omega t} - i\psi_{n+1} e^{-i(n+\frac{3}{2})\omega t}\right)$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\, i\sqrt{n+1}\left(e^{i\omega t} - e^{-i\omega t}\right)$$

$$= -\underbrace{\sqrt{\frac{\hbar}{2m\omega}}\sqrt{n+1}}_{\text{Amplitude}} \sin(\omega t)$$

(d) Compute the expectation value of the energy for $\psi(x,t)$.

$$\langle E \rangle = \sum_n |C_n(t)|^2 E_n = \frac{1}{2}\hbar\omega\left(n+\frac{1}{2}\right) + \frac{1}{2}\hbar\omega\left(n+\frac{3}{2}\right)$$

$$= \hbar\omega(n+1)$$

(e) Classically the energy of a harmonic oscillator is equal to $\frac{1}{2}kx_o^2 = \frac{1}{2}m\omega^2 x_o^2$, where x_o is the classical amplitude of oscillation in x. Using the energy of part (d), compute what the classical amplitude, x_o, would be. Compare x_o to the your result for the amplitude in $\langle x \rangle$ from part (c). Why do you think they differ?

$$\frac{1}{2}m\omega^2\left(\sqrt{\frac{\hbar}{2m\omega}}\sqrt{n+1}\right)^2 = \frac{1}{2}m\omega^2\frac{\hbar}{2m\omega}(n+1)$$

$$= \frac{1}{4}\hbar\omega(n+1)$$

The amplitude of $\langle x \rangle$ is not the classical amplitude of oscillation.

4. **Piecewise constant potentials**
For this problem consider the one dimensional time independent Schrodinger question with potential $V(x)$:

$$\begin{aligned} V(x) &= V_o \text{ for } x < 0 \\ V(x) &= -V_o \text{ for } x > 0 \end{aligned}$$

for $E > V_o > 0$.

(a) What is the general form of the solution for $x > 0$ and for $x < 0$? Remember $E > V_o > 0$. Make sure to define all the variables that you introduce.

$$\psi(x<0) = A_1 e^{ik_1 x} + A_1' e^{-ik_1 x} \qquad k_1 = \sqrt{\frac{2m(E-V_o)}{\hbar^2}}$$

$$\psi(x>0) = A_2 e^{ik_2 x} + A_2' e^{-ik_2 x} \qquad k_2 = \sqrt{\frac{2m(E+V_o)}{\hbar^2}}$$

(b) What are the boundary conditions at $x = 0$ expressed in terms of your wave functions from part (a)?

$$\psi(0) = A_1 + A_1' = A_2 + A_2'$$

$$\psi'(0) = ik_1(A_1 - A_1') = ik_2(A_2 - A_2')$$

(c) For a wave **coming from the right and going to the left**, which of the terms in part (a) is zero?

$A_1 = 0$

(d) Solve for the transmission and reflection probabilities for a wave coming from the right and going to the left.

$$\left.\begin{aligned} A_1' &= A_2 + A_2' \\ -A_1' &= \frac{k_2}{k_1}(A_2 - A_2') \end{aligned}\right\} \rightarrow 0 = \left(1+\frac{k_2}{k_1}\right)A_2 + \left(1-\frac{k_2}{k_1}\right)A_2'$$

$$\frac{A_2}{A_2'} = -\frac{1-k_2/k_1}{1+k_2/k_1} = -\frac{k_1-k_2}{k_1+k_2}$$

$$R = \left|\frac{A_2}{A_2'}\right|^2 = \boxed{\left(\frac{1-k_2/k_1}{1+k_2/k_1}\right)^2 = R}$$

$$\left(\frac{k_1-k_2}{k_1+k_2}\right)^2$$

$$\frac{A_1'}{A_2'} = \frac{A_2}{A_2'} + 1 = \frac{2k_2/k_1}{1+k_2/k_1} = \frac{2k_2}{k_1+k_2}$$

$$T = \frac{k_1}{k_2}\left|\frac{A_1'}{A_2'}\right|^2 = \frac{4k_2/k_1}{(1+k_2/k_1)^2} = T = \frac{4k_1k_2}{(k_1+k_2)^2}$$

(e) For what value of E/V_o is the transmission probability equal to 3/4?

$$\frac{3}{4} = \frac{4(k_2/k_1)}{(1+k_2/k_1)^2} \rightarrow 3\left(1+2\frac{k_2}{k_1}+\left(\frac{k_2}{k_1}\right)^2\right) = 16\left(\frac{k_2}{k_1}\right)$$

$$\rightarrow 0 = 3\left(\frac{k_2}{k_1}\right)^2 - 10\left(\frac{k_2}{k}\right) + 3$$

$$\rightarrow \frac{k_2}{k_1} = \frac{10 \pm \sqrt{100-36}}{6}$$

$$= \frac{10 \pm 8}{6} = 3 \text{ or } 1/3 \quad \text{since } \frac{k_2}{k_1} > 1$$

$$\rightarrow \left(\frac{k_2}{k_1}\right)^2 = \frac{E+V_o}{E-V_o} = 9 \rightarrow 9E - 9V_o = E + V_o$$

$$8E - 10V_o = 0$$

$$\boxed{\frac{E}{V_o} = \frac{10}{8} = \frac{5}{4} = 1.25}$$