Name:

Exam 2 - PHY 4604 - Fall 2017

November 15, 2017
8:20-10:20PM, NPB 1002

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

Unless otherwise noted, all parts (a), (b), ... are worth 5 points, and the entire exam is 100 points.

$$
\begin{aligned}
Y_0^0 &= \frac{1}{2}\sqrt{\frac{1}{\pi}} \\
Y_1^1(\theta,\varphi) &= \frac{-1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta \\
Y_1^0(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{\pi}}\cdot\cos\theta \\
Y_1^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta \\
Y_2^2(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot e^{2i\varphi}\cdot\sin^2\theta \\
Y_2^1(\theta,\varphi) &= \frac{-1}{2}\sqrt{\frac{15}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta\cdot\cos\theta \\
Y_2^0(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{5}{\pi}}\cdot(3\cos^2\theta-1) \\
Y_2^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{15}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta\cdot\cos\theta \\
Y_2^{-2}(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot e^{-2i\varphi}\cdot\sin^2\theta
\end{aligned}
$$

1. **Short Answer Section**

(a) What is the condition for a set of states to be complete in the Dirac bra-ket notation?

$$\mathbb{1} = \sum_n |\psi_n\rangle\langle\psi_n|$$

(b) What is the definition of the adjoint, $A^\dagger$, of an operator A?

$$\langle g | A f \rangle = \langle A^\dagger g | f \rangle$$

(c) What is the generalized uncertainty relation for two operators, A, and B, that do not commute?

For a particular $|\psi\rangle$

$$\Delta A \, \Delta B \geqslant \left| \frac{i}{2} \langle \psi | [A, B] | \psi \rangle \right|$$

(d) What are the allowed values of the angular momentum quantum numbers, l, for the Hydrogen atom eignenstates with energy $-13.6eV/n^2$?

$$l = 0, 1, 2, \ldots, n-1$$

2. **General properties**

Consider the Hamiltonian

$$H = \begin{pmatrix} 0 & E_o & 0 \\ E_o & 0 & E_o \\ 0 & E_o & 0 \end{pmatrix}. \tag{1}$$

(a) What are the eigenvalues of H?

$$\det\begin{pmatrix} -\lambda & E_o & 0 \\ E_o & -\lambda & E_o \\ 0 & E_o & -\lambda \end{pmatrix} = 0 = -\lambda^3 + 2E_o^2\lambda$$

$$= -\lambda(\lambda^2 - 2E_o)$$

$$\rightarrow \lambda = 0, +\sqrt{2}E_o, -\sqrt{2}E_o$$

(b) What are the eigenvectors for these eigenvalues?

$\lambda = 0:$ $\begin{pmatrix} 0 & E_o & 0 \\ E_o & 0 & E_o \\ 0 & E_o & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = 0$ $\quad C_2 = 0$, $C_1 = -C_3$ $\quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \equiv |\psi_1\rangle$

$\lambda = +\sqrt{2}E_o:$ $E_o\begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = 0$ $\quad \sqrt{2}C_1 = C_2$, $\sqrt{2}C_3 = C_2$ $\quad \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \equiv |\psi_2\rangle$

$\lambda = -\sqrt{2}E_o$ $\quad E_o\begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = 0$ $\quad -\sqrt{2}C_1 = C_2$, $-\sqrt{2}C_3 = C_2$ $\quad \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \equiv |\psi_3$

(c) Consider the operator

$$P_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2)$$

Prove whether or not it is possible to find simultaneous eigenvectors of P_{12} and H. If there are simultaneous eigenvectors, list them.

$$HP_{12} = E_0 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P_{12}H = E_0 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = E_0 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$\rightarrow [H, P_{12}] \neq 0 \rightarrow$ no simultaneous eigenvectors.

(d) Finally consider the operator

$$P_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (3)$$

Prove whether or not it is possible to find simultaneous eigenvectors of P_{13} and H. If there are simultaneous eigenvectors, list them.

$$HP_{13} = E_0 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = E_0 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$P_{13}H = E_0 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = E_0 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$\rightarrow [H, P_{13}] = 0 \rightarrow$ simultaneous eigevectors

From (b), $|\psi_1\rangle$: $E = 0$ and $\lambda = -1$ for P_{13}

$|\psi_2\rangle$: $E = +\sqrt{2}\,E_0 4$ & $\lambda = +1$ for P_{13}

$|\psi_3\rangle$: $E = -\sqrt{2}\,E_0$ & $\lambda = +1$ for P_{13}

3. **Measurements**

The eigenvectors and eigenvalues of hamiltonian are

$$|\psi_a\rangle = \frac{1}{2}(|1\rangle + |2\rangle + |3\rangle + |4\rangle) \text{ with } E_a = E_o \tag{4}$$

$$|\psi_b\rangle = \frac{1}{2}(|1\rangle + i|2\rangle - |3\rangle - i|4\rangle) \text{ with } E_b = 0 \tag{5}$$

$$|\psi_c\rangle = \frac{1}{2}(|1\rangle - i|2\rangle - |3\rangle + i|4\rangle) \text{ with } E_c = 0 \tag{6}$$

$$|\psi_d\rangle = \frac{1}{2}(|1\rangle - |2\rangle + |3\rangle - |4\rangle) \text{ with } E_d = -E_o, \tag{7}$$

where $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$ are an orthonormal basis of the system.

(a) Suppose the state of the system is $\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$ right before an energy measurement is made. What are possible outcomes of the energy measurement and their associated probabilities?

$$\left|\langle\psi_a|\tfrac{1}{\sqrt{2}}(|1\rangle+|2\rangle)\right|^2 = \left|\frac{2}{2\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$\left|\langle\psi_b|\tfrac{1}{\sqrt{2}}(|1\rangle+|2\rangle)\right|^2 = \left|\frac{1-i}{2\sqrt{2}}\right|^2 = \frac{2}{8}$$

$$\left|\langle\psi_c|\tfrac{1}{\sqrt{2}}(|1\rangle+|2\rangle)\right|^2 = \left|\frac{1+i}{2\sqrt{2}}\right|^2 = \frac{2}{8}$$

$$\left|\langle\psi_d|\tfrac{1}{\sqrt{2}}(|1\rangle+|2\rangle)\right|^2 = \left|\frac{0}{2\sqrt{2}}\right|^2 = 0$$

outcome	prob.
E_o	$1/2$
0	$1/2$

(b) Suppose the state of the system is at $t = 0$ is $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$, but instead of making an energy measurment right away, the system evolves in time (without making an energy measurement). What is the state of the system at time t, $|\psi(t)\rangle$?

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|\psi_a\rangle + \frac{1-i}{2\sqrt{2}}|\psi_b\rangle + \frac{1+i}{2\sqrt{2}}|\psi_c\rangle$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-iE_ot}|\psi_a\rangle + \frac{1-i}{2\sqrt{2}}|\psi_b\rangle + \frac{1+i}{2\sqrt{2}}|\psi_c\rangle$$

(c) An energy measurment is now made using $|\psi(t)\rangle$ from part (b). What are the possible outcomes and their associated probabilities?

Same as for (a).

(d) Suppose that the energy measured is E_o. What is the state of the system immediately after the measurement?

$|\psi_a\rangle$

4. **Angular momentum**

(a) What is the commutator $[L_+^2, Lz]$?

$$[L_+, L_z] = [L_x + iL_y, L_z] = i\hbar(-L_y + iL_x)$$
$$= \hbar(-L_x - iL_y) = -\hbar L_+$$

$$[L_+^2, L_z] = L_+^2 L_z - L_+L_zL_+ + L_+L_zL_+ - L_zL_+^2$$
$$= L_+[L_+, L_z] + [L_+, L_z]L_+$$
$$= -2\hbar L_+^2 = -2\hbar(L_x + iL_y)^2$$

(b) What is the matrix element $\langle \frac{3}{2}, \frac{3}{2}|L_+^2|\frac{3}{2}, -\frac{1}{2}\rangle$?

$$L_+ |\tfrac{3}{2}, -\tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1) - -\tfrac{1}{2}(-\tfrac{1}{2}+1)}\ |\tfrac{3}{2}, \tfrac{1}{2}\rangle$$
$$= \hbar\sqrt{\frac{15+1}{4}}\ |\tfrac{3}{2}, \tfrac{1}{2}\rangle = 2\hbar\ |\tfrac{3}{2}, \tfrac{1}{2}\rangle$$

$$L_+ |\tfrac{3}{2}, \tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)}\ |\tfrac{3}{2}, \tfrac{3}{2}\rangle$$
$$= \hbar\sqrt{\frac{15-3}{4}}\ |\tfrac{3}{2}, \tfrac{3}{2}\rangle = \hbar\sqrt{3}\ |\tfrac{3}{2}, \tfrac{3}{2}\rangle = \hbar\frac{3}{\sqrt{2}}|\tfrac{3}{2}, \tfrac{3}{2}\rangle$$

$$\rightarrow \langle \tfrac{3}{2}, \tfrac{3}{2}|L_+^2|\tfrac{3}{2}, -\tfrac{1}{2}\rangle = (2\hbar)(\sqrt{3}\hbar) = \underbrace{2\sqrt{3}}_{\sqrt{12}}\ \hbar^2$$

(c) Evaluate the non-zero matrix elements of $\langle \frac{3}{2}, m | L_z | \frac{3}{2}, m' \rangle$.

$$\langle \tfrac{3}{2}, \tfrac{3}{2} | L_z | \tfrac{3}{2}, \tfrac{3}{2} \rangle = \hbar \tfrac{3}{2}$$

$$\langle \tfrac{3}{2}, \tfrac{1}{2} | L_z | \tfrac{3}{2}, \tfrac{1}{2} \rangle = \hbar \cdot \tfrac{1}{2}$$

$$\langle \tfrac{3}{2}, \tfrac{-1}{2} | L_z | \tfrac{3}{2}, \tfrac{-1}{2} \rangle = -\tfrac{\hbar}{2}$$

$$\langle \tfrac{3}{2}, \tfrac{-3}{2} | L_z | \tfrac{3}{2}, \tfrac{-3}{2} \rangle = -\tfrac{3}{2}\hbar$$

(d) In spherical coordinates a wave function has the form $\psi(r, \theta, \phi) = R(r) cos^2(\theta)$. If a total angular momentum, L^2, measurement is performed on this state, what are the possible outcomes and their associated probabilities?

$$Y_2^0(\theta, \varphi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(3\cos^2\theta - 1\right)$$

$$Y_0^0(\theta, \varphi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

$$\rightarrow Y_2^0(\theta, \varphi) + \frac{\sqrt{5}}{2} Y_0^0(\theta, \varphi) \propto \cos^2\theta$$

$$\int d\Omega \left| Y_2^0 + \frac{\sqrt{5}}{2} Y_0^0 \right|^2 = 1 + \frac{5}{4} = \frac{9}{4}$$

$$\rightarrow \frac{2}{3}\left(Y_2^0 + \frac{\sqrt{5}}{2} Y_0^0\right) = \frac{2}{3} Y_2^0 + \frac{\sqrt{5}}{3} Y_0^0 \propto \cos^2\theta$$

$\rightarrow$

outcome	prob.
$\ell = 0$	5/9
$\ell = 2$	4/9

5. **Radial Schrodinger equation**

A particle moves in the radial potential given by $V(r) = -V_o$ for $r < a$ and $V(r) = V_o$ for $r > a$ with $V_o > 0$. In the following take the energy to be in the range $-V_o < E < V_o$, and take the angular momentum to be zero, $l = 0$.

(a) What is the solution to the $u(r)$ radial equation for $r < a$ with $-V_o < E < V_o$ and $l = 0$? Eliminate any unphysical solutions as $r \to 0$.

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} - V_o u = Eu$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = (E+V_o)u$$

$$k = \sqrt{\frac{2m}{\hbar^2}(E+V_o)}$$

$$\frac{d^2u}{dr^2} = -\frac{2m}{\hbar^2}\underbrace{(E+V_o)}_{>0}u \rightarrow u(r) = A\sin(kr)$$

since $\cos(kr)$ is not physical. It implies $R(r) \propto \frac{1}{r}$ as $r \to 0$.

(b) What is the solution to the $u(r)$ radial equation for $r > a$ with $-V_o < E < V_o$ and $l = 0$? Eliminate any unphysical solutions as $r \to \infty$.

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V_o u = Eu$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = (E-V_o)u$$

$$\frac{d^2u}{dr^2} = \frac{2m}{\hbar^2}\underbrace{(V_o-E)}_{>0}u \qquad K = \sqrt{\frac{2m}{\hbar^2}(V_o-E)}$$

$$u(r) = Be^{-Kr}$$

since e^{+Kr} is not normalizable

(c) Derive the boundary conditions for these $u(r)$ at $r = a$.

$$u(a) = A\sin(ka) = Be^{-Ka}$$

$$u'(a) = kA\cos(ka) = -KBe^{-Ka}$$

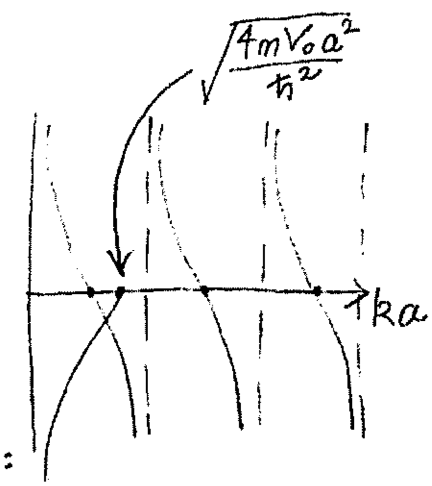
$$\rightarrow k\cot(ka) = -K$$

$$\cot(ka) = -\frac{K}{k} \quad \text{or} \quad \tan(ka) = -\frac{k}{K}$$

(d) Solve the boundary conditions graphically. What is the minimum value of V_o so that there is at least one solution for $-V_o < E < V_o$?

$$k^2 + K^2 = \frac{2mV_0}{\hbar^2} + \frac{2mV_0}{\hbar^2} = \frac{4mV_0}{\hbar^2}$$

$$\cot(ka) = -\frac{\sqrt{\frac{4mV_0a^2}{\hbar^2} - (ka)^2}}{ka}$$



To have at least one solution:

$$\sqrt{\frac{4mV_0a^2}{\hbar^2}} > \frac{\pi}{2}$$

$$\frac{4mV_0a^2}{\hbar^2} > \frac{\pi^2}{4}$$

$$V_0 > \frac{\hbar^2\pi^2}{16ma^2}.$$