Schrodinger Eq. in 3D:

$$H = \frac{p^2}{2m} + V(r)$$

$$= \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + V(r)$$

$$= \frac{1}{2m}\left(\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 + \left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right)^2 + \left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right)^2\right) + V(r)$$

$$= \frac{-\hbar^2}{2m}\nabla^2 + V(r)$$

For a spherically symmetric problem $V(r) = V(|r|)$. Write ∇^2 in spherical coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)$$
$$+ \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2}\right),$$

where $x = r\sin\theta\cos\varphi$.
$y = r\sin\theta\sin\varphi$
$z = r\cos\theta$

We are going to solve

$$H\psi = \left(\frac{-\hbar^2}{2m}\nabla^2 + V(|r|)\right)\psi(r,\theta,\varphi) = E\,\psi(r,\theta,\varphi).$$

First introduce the angular momentum operator:

$$\vec{L} = \vec{r}\times\vec{p}$$
$$L_x = y p_z - z p_y$$
$$L_y = z p_x - x p_z$$
$$L_z = x p_y - y p_x$$

$\vec{L}$ is Hermitian:

$$\begin{aligned}(L_x)^\dagger &= (y p_z - z p_y)^\dagger \\ &= (y p_z)^\dagger - (z p_y)^\dagger \\ &= p_z^\dagger y^\dagger - p_y^\dagger z^\dagger \\ &= p_z y - p_y z \\ &= y p_z - z p_y = L_x\end{aligned}$$

In spherical coordinates

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

and $L_z = \dfrac{\hbar}{i}\dfrac{\partial}{\partial\varphi}$.

$$\rightarrow \quad -\frac{\hbar^2 \nabla^2}{2m} = -\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{2mr^2} L^2$$

$$\rightarrow \quad \boxed{H = -\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{L^2}{2mr^2} + V(|r|)}$$

Note:

$[H, L^2] = 0$ because L^2 has no r dependence and $[L^2, L^2] = 0$

$[H, L_z] = 0$ because L_z has no r dependence and $[L_z, L^2] = 0$

$[L^2, L_z] = 0$ because $\frac{\partial}{\partial \varphi}$ commutes with L^2.

$$\boxed{\text{This means we can find simultaneous eigenvectors of } H, L^2, L_z.}$$

We are first going to find the eigenvectors of L^2, L_z.