Addition of Angular Momentum

Take two spin 1/2 particles, e.g. e^- & p of hydrogen atom. (distinguishable)

Four possibilities: $\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$

Total angular momentum: $\vec{S} = \vec{S}^{(1)} + \vec{S}^{(2)}$

$$[S_x, S_y] = [S_x^{(1)} + S_x^{(2)}, S_y^{(1)} + S_y^{(2)}]$$

$$= [S_x^{(1)}, S_y^{(1)}] + [S_x^{(2)}, S_y^{(2)}]$$

$$= i\hbar S_z^{(1)} + i\hbar S_z^{(2)}$$

$$= i\hbar S_z$$

spin state

$$S_z \chi_1 \chi_2 = (S_z^{(1)} + S_z^{(2)}) \chi_1 \chi_2$$

$$= (S_z^{(1)} \chi_1) \chi_2 + \chi_1 (S_z^{(2)} \chi_2)$$

$$= \hbar m_1 \chi_1 \chi_2 + \chi_1 \hbar m_2 \chi_2$$

$$= \hbar \underbrace{(m_1 + m_2)}_{m} \chi_1 \chi_2$$

$\chi_1 \; \chi_2$

$\uparrow \; \uparrow \; : \; m = 1$

$\uparrow \; \downarrow \; : \; m = 0$

$\downarrow \; \uparrow \; : \; m = 0$

$\downarrow \; \downarrow \; : \; m = -1$ What about the total $\vec{J}$?

$$S^2 = (S_x^{(1)} + S_x^{(2)})^2 + (S_y^{(1)} + S_y^{(2)})^2 + (S_z^{(1)} + S_z^{(2)})^2$$

$$= (S^{(1)})^2 + (S^{(2)})^2 + S^{(1)} \cdot S^{(2)} + S^{(2)} \cdot S^{(1)}$$

$$= (S^{(1)})^2 + (S^{(2)})^2 + 2\, S^{(1)} \cdot S^{(2)}$$

$$S^{(1)} \cdot S^{(2)} = S_x^{(1)} S_x^{(2)} + S_y^{(1)} S_y^{(2)} + S_z^{(1)} S_z^{(2)}$$

$$= \frac{1}{2} (S_x^{(1)} + i S_y^{(1)})(S_x^{(2)} - i S_y^{(2)})$$

$$+ \frac{1}{2} (S_x^{(1)} - i S_y^{(1)})(S_x^{(2)} + i S_y^{(2)})$$

$$+ S_z^{(1)} S_z^{(2)}$$

$\rightarrow$ $$S^{(1)} \cdot S^{(2)} = \frac{1}{2}\left(S_+^{(1)} S_-^{(2)} + S_-^{(1)} S_+^{(2)}\right) + S_z^{(1)} S_z^{(2)}$$

$\rightarrow$ $$S^2 = (S^{(1)})^2 + (S^{(2)})^2 + S_+^{(1)} S_-^{(2)} + S_-^{(1)} S_+^{(2)} + 2 S_z^{(1)} S_z^{(2)}$$

$$S^2|\uparrow\uparrow\rangle = (S^{(1)})^2|\uparrow\uparrow\rangle + (S^{(2)})^2|\uparrow\uparrow\rangle + 2S_z^{(1)}S_z^{(2)}|\uparrow\uparrow\rangle$$

$$= \hbar^2\tfrac{1}{2}(\tfrac{1}{2}+1)|\uparrow\uparrow\rangle + \hbar^2\tfrac{1}{2}(\tfrac{1}{2}+1)|\uparrow\uparrow\rangle + 2(\hbar/2)^2|\uparrow\uparrow\rangle$$

$$= \hbar^2\left(\frac{3}{4}+\frac{3}{4}+\frac{1}{2}\right)|\uparrow\uparrow\rangle$$
$$= \hbar^2 2|\uparrow\uparrow\rangle$$
$$= \hbar^2 1(1+1)|\uparrow\uparrow\rangle \quad \Rightarrow j=1$$

$$\boxed{|\underset{j}{1},\underset{m}{1}\rangle = |\uparrow\uparrow\rangle}$$

Apply lowering operator:

$$S_-|1,1\rangle = \hbar\sqrt{1(1+1)-1(1-1)}\,|1,0\rangle = \hbar\sqrt{2}\,|1,0\rangle$$

$$= (S_-^{(1)} + S_-^{(2)})|\uparrow\uparrow\rangle$$

$$= \hbar\underbrace{\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}}_{1}\,|\downarrow\uparrow\rangle + \hbar|\uparrow\downarrow\rangle$$

$$\Rightarrow \boxed{|1,0\rangle = \frac{1}{\sqrt{2}}\left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right)}$$

$$S_-|1,0\rangle = \hbar\sqrt{1(1+1)-0(0-1)}\,|1,-1\rangle = \hbar\sqrt{2}\,|1,-1\rangle$$

$$= (S_-^{(1)} + S_-^{(2)})\frac{1}{\sqrt{2}}\left(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(S_-^{(2)}|\downarrow\uparrow\rangle + S_-^{(1)}|\uparrow\downarrow\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|\downarrow\downarrow\rangle + |\downarrow\downarrow\rangle\right) = \sqrt{2}\,|\downarrow\downarrow\rangle$$

$$\rightarrow \boxed{|1,-1\rangle = |\downarrow\downarrow\rangle}$$

What about the other $m=0$ combination?

$$S^2\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) =$$

$$= \left((S^{(1)})^2 + (S^{(2)})^2 + S_+^{(1)}S_-^{(2)} + S_-^{(1)}S_+^{(2)} + 2S_z^{(1)}S_z^{(2)}\right)$$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$= \left(\hbar^2\frac{1}{2}\left(\frac{1}{2}+1\right) + \hbar^2\frac{1}{2}\left(\frac{1}{2}+1\right) - \hbar^2 - 2\left(\frac{\hbar}{2}\right)^2\right)$$

$$\frac{1}{2}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$= 0 \quad \Rightarrow j = 0 \Rightarrow \boxed{|0,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)}$$