Solution

Name:

Exam 1 - PHY 4604 - Fall 2018
October 3, 2018
8:20-10:10PM, NPB 1002

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

Unless otherwise noted all parts (a), (b), ... are worth 5 points, and the entire exam is 100 points.

Harmonic oscillator:

$$\begin{aligned}
a_+ &= \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x) \\
a_- &= \frac{1}{\sqrt{2\hbar m\omega}}(+ip + m\omega x) \\
x &= \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \\
p &= i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-) \\
\psi_0(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\
\psi_n &= \frac{1}{\sqrt{n!}}\,(a_+)^n\,\psi_0
\end{aligned}$$

1. **Short answer section**

(a) Write down the time dependent Schrodinger equation in one dimension.

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi$$

(b) What are the normalized solutions to the time independent Schrodinger equation for an infinite square between $0 \le x \le a$?

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi n x}{a}\right)$$

(c) Let $H\phi_n(x) = E_n\phi_n(x)$ be the solutions to the time independent Schrodinger equation. Any function $\psi(x)$ may be written as

$$\psi(x) = \sum_n c_n \phi_n(x). \tag{1}$$

What are the c_n in terms of $\psi(x)$ and the $\phi_n(x)$?

$$c_n = \int \phi_n^*(x)\psi(x)\,dx$$

(d) What is the solution to the time independent Schrodinger equation with a constant potential $V(x) = V_o$ and $E < V_o$?

$$\psi(x) = Ae^{\rho x} + Be^{-\rho x}, \quad \rho = \sqrt{\frac{2m(V_o - E)}{\hbar^2}}$$

(e) What is the commutator $[x, p]$?

$$[x, p] = i\hbar$$

2. **General properties**

Consider the wave function $\psi(x) = C\,x(1-x)$ on the interval $0 \le x \le 1$.

(a) What is the constant C so that the wave function is normalized?

$$1 = \int_0^1 C^2 x^2 (1 - 2x + x^2)\,dx$$

$$= C^2 \int_0^1 (x^2 - 2x^3 + x^4)\,dx$$

$$= C^2 \cdot \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) = C^2 \left(\frac{20 - 30 + 12}{60}\right) = C^2 \frac{2}{60}$$

$$C = \sqrt{30}$$

(b) Compute the expectation values $\langle x \rangle$, $\langle x^2 \rangle$, as well as σ_x for $\psi(x, 0)$.

$$\langle x \rangle = \frac{1}{2}$$

$$\langle x^2 \rangle = C^2 \int_0^1 (x^4 - 2x^5 + x^6)\,dx$$

$$= \underbrace{C^2}_{30} \left(\frac{1}{5} - \frac{2}{6} + \frac{1}{7}\right)$$

$$= 6 - 10 + \frac{30}{7} = \frac{30}{7} - \frac{28}{7} = \frac{2}{7}$$

$$\sigma_x = \sqrt{\frac{2}{7} - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{28}} = \frac{1}{2}\frac{1}{\sqrt{7}}$$

(c) Compute the expectation values $\langle p \rangle$, $\langle p^2 \rangle$, as well as σ_p for $\psi(x, 0)$.

$$\langle p \rangle = 0$$

$$p^2 \, x(1-x) = +2\hbar^2$$

$$\rightarrow \langle p^2 \rangle = c^2 \cdot 2 \cdot \hbar^2 \int_0^1 x(1-x)\,dx$$

$$= 30 \cdot 2\hbar^2 \cdot \underbrace{\left(\frac{1}{2} - \frac{1}{3}\right)}_{1/6} = 10\,\hbar^2$$

$$\sigma_p = \sqrt{10}\,\hbar$$

(d) Use the results of (b) and (c) above to check that the uncertainty principle is satisfied.

$$\sigma_x \sigma_p = \frac{1}{2}\frac{1}{\sqrt{7}}\sqrt{10}\,\hbar = \sqrt{\frac{5}{14}}\,\hbar > \sqrt{\frac{5}{20}}\,\hbar = \frac{\hbar}{2}$$

(e) What is the commutator, $[p^2, x^2]$?

$$= p^2 x^2 - p x^2 p + p x^2 p - x^2 p^2$$

$$= p[p, x^2] + [p, x^2]\,p$$

but $[p, x^2] = p x^2 - x p x + x p x - x^2 p$

$$= [p, x]x + x[p, x]$$

$$= -2i\hbar x$$

$$\rightarrow [p^2, x^2] = -2i\hbar\,(\underbrace{p x}_{i\hbar} + x p)$$

Also $= -2i\hbar\,(2px + \overbrace{[x, p]}^{i\hbar}) = 2\hbar^2 - 4i\hbar\,px = 2\hbar^2\left(1 - 2\frac{\partial}{\partial x}x\right)$

and $= -2i\hbar\,(\underbrace{[p, x]}_{-i\hbar} + 2xp)^4 = -2\hbar^2 - 4i\hbar\,xp = -2\hbar^2\left(1 + 2x\frac{\partial}{\partial x}\right)$

3. **Harmonic oscillator**

At $t = 0$ the wave function of a particle in a harmonic oscillator potential is given by

$$\psi(x,0) = \frac{i}{\sqrt{3}}\psi_1(x) + \frac{\sqrt{2}}{\sqrt{3}}\psi_3(x).$$

(a) What is $\psi(x,t)$?

$$\psi(x,t) = \frac{i}{\sqrt{3}}\psi_1(x)e^{-i\frac{3}{2}\omega t} + \frac{\sqrt{2}}{\sqrt{3}}\psi_3(x)e^{-i\frac{7}{2}\omega t}$$

(b) What are the expectation value of $\langle x\rangle$ and $\langle x^2\rangle$ as a function of time?

$$\langle x\rangle = 0$$

$$x^2 = \frac{\hbar}{2m\omega}\left(\overbrace{a_+a_+}^{1\to 3} + \overbrace{a_+a_- + a_-a_+}^{1\to 1 \text{ or } 3\to 3} + \overbrace{a_-a_-}^{3\to 1}\right)$$

$$\langle x^2\rangle = \int dx\left(\frac{-i}{\sqrt{3}}\psi_1^* e^{+i\frac{3}{2}\omega t} + \sqrt{\frac{2}{3}}\,\psi_3^* e^{i\frac{7}{2}\omega t}\right)$$

$$x^2\left(\frac{i}{\sqrt{3}}\psi_1 e^{-i\frac{3}{2}\omega t} + \sqrt{\frac{2}{3}}\,\psi_3 e^{-i\frac{7}{2}\omega t}\right)$$

$$= \frac{\hbar}{2m\omega}\left(\frac{i}{\sqrt{3}}\sqrt{\frac{2}{\cancel{3}}}\sqrt{2}\cancel{\sqrt{3}}\,e^{i2\omega t} - \frac{i}{\sqrt{3}}\sqrt{\frac{2}{\cancel{3}}}\cancel{\sqrt{3}}\sqrt{2}\,e^{-2i\omega t}\right.$$

$$\left. + \frac{1}{3}\cdot 3 + \frac{2}{3}\cdot 7\right)$$

$$= \frac{\hbar}{2m\omega}\left(\frac{17}{3} - \frac{4}{\sqrt{3}}\sin(2\omega t)\right)$$

(c) What are the expectation values of $\langle p\rangle$ and $\langle p^2\rangle$ as a function of time?

$$\langle p\rangle = 0$$

$$p^2 = \frac{\hbar m\omega}{2}\left(a_+a_- + a_-a_+ - a_+a_+ - a_-a_-\right)$$

$$\langle p^2\rangle = \frac{\hbar m\omega}{2}\left(\frac{17}{3} + \frac{4}{\sqrt{3}}\sin(2\omega t)\right)$$

(d) Compute σ_x, σ_p, and check that the uncertainty principle is satisfied.

$$\sigma_x = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{17}{3} - \frac{4}{\sqrt{3}}\sin(2\omega t)} \geqslant \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{17}{3} - \frac{4}{\sqrt{3}}}$$

$$\sigma_p = \sqrt{\frac{\hbar m\omega}{2}}\sqrt{\frac{17}{3} + \frac{4}{\sqrt{3}}\sin(2\omega t)} \geqslant \sqrt{\frac{\hbar m\omega}{2}}\sqrt{\frac{17}{3} - \frac{4}{\sqrt{3}}}$$

$$\sigma_x\sigma_p = \frac{\hbar}{2}\left(\frac{17}{3} - \frac{4}{\sqrt{3}}\right) > \frac{\hbar}{2}$$

(e) What is the expectation value of the energy for $\psi(x,t)$?

$$H = \hbar\omega\left(a_+a_- + \frac{1}{2}\right); \quad H\psi_n = \hbar\omega\left(n+\frac{1}{2}\right)\psi_n$$

$$\langle E\rangle = \langle H\rangle = \hbar\omega\left(\frac{1}{3}\left(1+\frac{1}{2}\right) + \frac{2}{3}\left(3+\frac{1}{2}\right)\right)$$

$$= \hbar\omega\left(\frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{2} + \frac{\cancel{2}}{3}\cdot\frac{7}{\cancel{2}}\right) = \hbar\omega\left(\frac{17}{6}\right)$$

4. **Piecewise constant potentials**
For this problem consider the one dimensional time independent Schrodinger equation for $E > V_o > 0$ with potential $V(x)$:

$$\begin{aligned} V(x) &= 0 \text{ for } x < 0 \\ V(x) &= V_o \text{ for } 0 < x < L \\ V(x) &= V_o/2 \text{ for } x > L. \end{aligned}$$

(a) The solution to the time independent Schrodinger equation in the regions $x < 0$ and $x > L$ have the form

$$\begin{aligned} \psi(x < 0) &= A_1 e^{ik_1 x} + A_1' e^{-ik_1 x} \\ \psi(x > L) &= A_3 e^{ik_3 x} + A_3' e^{-ik_3 x}. \end{aligned}$$

What are k_1 and k_3 in terms of E, V_o, $\hbar$, and m? What is the form of the solution for $0 < x < L$, including the wave vector?

$\psi(0<x<L) = A_2 e^{ik_2 x} + A_2' e^{-ik_2 x}$

$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$, $k_2 = \sqrt{\frac{2m(E-V_o)}{\hbar^2}}$, $k_3 = \sqrt{\frac{2m(E-V_o/2)}{\hbar^2}}$

(b) For a wave coming from the left and going to the right, which of the terms in part (a) is zero?

$A_3' = 0$

(c) What are the boundary conditions at $x = 0$ and at $x = L$ expressed in terms of the wave functions from part (a)?

$$\psi(0) = A_1 + A_1' = A_2 + A_2'$$

$$\psi'(0) = ik_1(A_1 - A_1') = ik_2(A_2 - A_2')$$

$$\psi(L) = (A_2 e^{ik_2L} + A_2' e^{-ik_2L}) = (A_3 e^{ik_3L} + A_3' e^{-ik_3L})$$

$$\psi'(L) = ik_2 (A_2 e^{ik_2L} - A_3' e^{-ik_2L}) = ik_3 (A_3 e^{ik_3L} - A_3' e^{-ik_3L})$$

(d) Using the probability current, derive an expression for the transmission and reflection probabilities for the case of part (b).

$$j = \frac{\hbar k_1}{m}|A_1|^2 - \frac{\hbar k_1}{m}|A_1'|^2 = \frac{\hbar k_3}{m}|A_3|^2$$

$$\longrightarrow \quad |A_1|^2 = |A_1'|^2 + \frac{k_3}{k_1}|A_3|^2$$

$$\longrightarrow \quad 1 = \underbrace{\frac{|A_1'|^2}{|A_1|^2}}_{R} + \underbrace{\frac{k_3}{k_1}\frac{|A_3|^2}{|A_1|^2}}_{T}$$

(e) After solving the boundary conditions in part (b), the coefficients are related by

$$\begin{pmatrix} A_1 \\ A_1' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} A_3 \\ A_3' \end{pmatrix}. \tag{2}$$

What are the transmission and reflection probabilities for the case of parts (c) and (d) in terms of k_1, k_3, and α, β, γ, δ?

Using $A_3' = 0$,

$$A_1 = \alpha A_3$$

$$A_1' = \gamma A_3$$

$$R = \frac{|A_1'|^2}{|A_1|^2} = \frac{|\gamma|^2}{|\alpha|^2}$$

$$T = \frac{k_3}{k_1} \frac{|A_3|^2}{|A_1|^2} = \frac{k_3}{k_1} \frac{1}{|\alpha|^2}$$