Name: Solution

Exam 2 - PHY 4604 - Fall 2018
November 14, 2018
8:20-10:10PM, NPB 1002

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

Unless otherwise noted, all parts (a), (b), ... are worth 5 points, and the entire exam is 100 points.

$$
\begin{aligned}
Y_0^0 &= \frac{1}{2}\sqrt{\frac{1}{\pi}} \\
Y_1^1(\theta,\varphi) &= \frac{-1}{2}\sqrt{\frac{3}{2\pi}} \cdot e^{i\varphi} \cdot \sin\theta \\
Y_1^0(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{\pi}} \cdot \cos\theta \\
Y_1^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot e^{-i\varphi} \cdot \sin\theta \\
Y_2^2(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot e^{2i\varphi} \cdot \sin^2\theta \\
Y_2^1(\theta,\varphi) &= \frac{-1}{2}\sqrt{\frac{15}{2\pi}} \cdot e^{i\varphi} \cdot \sin\theta \cdot \cos\theta \\
Y_2^0(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{5}{\pi}} \cdot (3\cos^2\theta - 1) \\
Y_2^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{15}{2\pi}} \cdot e^{-i\varphi} \cdot \sin\theta \cdot \cos\theta \\
Y_2^{-2}(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot e^{-2i\varphi} \cdot \sin^2\theta
\end{aligned}
$$

1. **Short Answer Section**

(a) Express a linear operator A as a matrix in the basis $|\psi_n\rangle$ for $n = 1, 2, 3, \ldots$.

$$\begin{pmatrix} \langle\psi_1|A|\psi_1\rangle & \langle\psi_1|A|\psi_2\rangle & \cdots \\ \langle\psi_2|A|\psi_1\rangle & \langle\psi_2|A|\psi_2\rangle & \\ \vdots & & \ddots \end{pmatrix}$$

(b) What is the the adjoint of the product of two operators, AB?

$$(AB)^\dagger = B^\dagger A^\dagger$$

(c) What is the commutator $[J_x, J_z]$?

$$[J_z, J_x] = i\hbar J_y \rightarrow [J_x, J_z] = -i\hbar J_y$$

(d) What are the allowed values of the angular momentum quantum numbers, l, for the Hydrogen atom eigenstates with energy $-13.6eV/n^2$?

$$l = 0, 1, 2, \ldots, n-1$$

2. **General properties**

Consider the operator

$$A = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}. \tag{1}$$

(a) What are the eigenvalues of A?

$$\det \begin{pmatrix} 2-\lambda & i \\ -i & 2-\lambda \end{pmatrix} = 0 = (2-\lambda)^2 - 1 = 0$$

$$\rightarrow \lambda - 2 = \pm 1$$

$$\lambda = 3, 1$$

(b) What are the eigenvectors for these eigenvalues?

$$\lambda = 3: \quad \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 3\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \rightarrow \begin{matrix} 2c_1 + i c_2 = 3c_1 \\ i c_2 = c_1 \end{matrix} \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} = |\psi_a\rangle$$

$$\lambda = 1: \quad \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 1\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \rightarrow \begin{matrix} 2c_1 + i c_2 = c_1 \\ c_1 + i c_2 = 0 \end{matrix} \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = |\psi_b\rangle$$

(c) Let the eigenvectors be denoted by $|\psi_a\rangle$ and $|\psi_b\rangle$. What are the projection operators $|\psi_a\rangle\langle\psi_a|$ and $|\psi_b\rangle\langle\psi_b|$?

$$\lambda = 3: \quad |\psi_a\rangle\langle\psi_a| = \frac{1}{2}\begin{pmatrix} i \\ 1 \end{pmatrix}\begin{pmatrix} -i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} i(-i) & i1 \\ 1(-i) & 11 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

$$\lambda = 1: \quad |\psi_b\rangle\langle\psi_b| = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}\begin{pmatrix} 1 & -i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 11 & 1(-i) \\ i1 & i(-i) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

(d) Define an operator B by

$$B = \begin{pmatrix} 1 & 3i \\ -3i & 1 \end{pmatrix}. \tag{2}$$

Prove whether or not it is possible to find simultaneous eigenvectors of B and A. If there are simultaneous eigenvectors, list them.

$$\begin{pmatrix} 1 & 3i \\ -3i & 1 \end{pmatrix}\begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix} = \begin{pmatrix} 5 & 7i \\ -7i & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}\begin{pmatrix} 1 & 3i \\ -3i & 1 \end{pmatrix} = \begin{pmatrix} 5 & 7i \\ -7i & 5 \end{pmatrix} \rightarrow [A, B] = 0$$

Since $|\psi_a\rangle$ & $|\psi_b\rangle$ are not degenerate, the simultaneous eigenvalues are $|\psi_a\rangle$ & $|\psi_b\rangle$.

3. **Measurements**

The eigenvectors and eigenvalues of Hamiltonian are

$$|\psi_a\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle) \text{ with } E_a = 0 \tag{3}$$

$$|\psi_b\rangle = \frac{1}{\sqrt{6}}(|1\rangle - 2|2\rangle + |3\rangle) \text{ with } E_b = +E_o \tag{4}$$

$$|\psi_c\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |3\rangle) \text{ with } E_c = -E_o, \tag{5}$$

where $|1\rangle$, $|2\rangle$, $|3\rangle$ are an orthonormal basis of the system.

(a) Suppose the state of the system is $\frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$ right before an energy measurement is made. What are possible outcomes of the energy measurement and their associated probabilities?

outcome	probability
0	$\lvert\langle\psi_a\vert\psi(0)\rangle\rvert^2 = \lvert 0\rvert^2$
$+E_o$	$\lvert\langle\psi_b\vert\psi(0)\rangle\rvert^2 = \left\lvert\frac{1}{\sqrt{6}}\cdot\frac{1}{\sqrt{2}}\cdot(1+2)\right\rvert^2 = \left\lvert\frac{\sqrt{3}}{2}\right\rvert^2 = \frac{3}{4}$
$-E_o$	$\lvert\langle\psi_c\vert\psi(0)\rangle\rvert^2 = \left\lvert\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\cdot 1\right\rvert^2 = \left\lvert\frac{1}{2}\right\rvert^2 = \frac{1}{4}$

(b) Suppose the state of the system is at $t = 0$ is $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$, but instead of making an energy measurement right away, the system evolves in time (without making an energy measurement). What is the state of the system at time t, $|\psi(t)\rangle$?

$$|\psi(t)\rangle = \frac{\sqrt{3}}{2} e^{-iE_o t/\hbar} |\psi_b\rangle + \frac{1}{2} e^{iE_o t/\hbar} |\psi_c\rangle$$

(c) At time t a position measurement is made with the operator

$$X = 1|1\rangle\langle 1| + 2|2\rangle\langle 2| + 3|3\rangle\langle 3|. \tag{6}$$

What are the possible outcomes and their associated probabilities?

<u>outcome</u> <u>probability</u>

1 $\quad |\langle 1|\psi(t)\rangle|^2 = \left| \frac{\sqrt{3}}{2} e^{-iE_0t/\hbar} \frac{1}{\sqrt{6}} + \frac{1}{2} e^{iE_0t/\hbar} \frac{1}{\sqrt{2}} \right|^2$

$$= \left| \frac{1}{\sqrt{2}} \cos(E_0t/\hbar) \right|^2 = \frac{1}{2} \cos^2(E_0t/\hbar)$$

2 $\quad |\langle 2|\psi(t)\rangle|^2 = \left| \frac{\sqrt{3}}{2} e^{-iE_0t/\hbar} \left(\frac{-2}{\sqrt{6}}\right) \right|^2 = \frac{1}{2}$

3 $\quad |\langle 3|\psi(t)\rangle|^2 = \left| \frac{\sqrt{3}}{2} e^{-iE_0t/\hbar} \frac{1}{\sqrt{6}} + \frac{1}{2} e^{iE_0t/\hbar} \left(\frac{-1}{\sqrt{2}}\right) \right|^2$

$$= \frac{1}{2} \sin^2(E_0t/\hbar)$$

(d) If steps (b) and (c) were repeated many times, what would the average value of the position measurements be?

$$\langle X \rangle = \frac{1}{2} \cos^2(E_0t/\hbar) + 2 \cdot \frac{1}{2} + \frac{3}{2} \sin^2(E_0t/\hbar)$$

$$= \frac{3}{2} + \sin^2(E_0t/\hbar)$$

4. **Angular momentum**

(a) What is the commutator $[L_y^2 + L_z^2, L_y + L_z]$?

$$= [L_y^2, L_z] + [L_z^2, L_y]$$

$$= L_y[L_y, L_z] + [L_y, L_z]L_y + L_z[L_z, L_y] + [L_z, L_y]L_z$$

$$= i\hbar\,(L_yL_x + L_xL_y - L_zL_x - L_xL_z)$$

(b) What is the matrix element $\langle 2,2|L_y^2|2,0\rangle$?

$$L_y = \frac{L_+ - L_-}{2i}$$

$$\rightarrow \langle 2,2|L_y^2|2,0\rangle = \frac{1}{(2i)^2}\langle 2,2|L_+^2|2,0\rangle$$

$$= -\frac{1}{4}\hbar\sqrt{2(2+1) - 0(0+1)}\,\langle 2,2|L_+|2,1\rangle$$

$$= -\frac{1}{4}\hbar\sqrt{6}\;\hbar\sqrt{2(2+1) - 1(1+1)}$$

$$= -\hbar^2\,\frac{2\sqrt{6}}{4} = -\hbar^2\sqrt{\frac{3}{2}}$$

(c) Evaluate the non-zero matrix elements of $\langle \frac{3}{2}, m|L_+|\frac{3}{2}, m'\rangle$.

$$\langle \tfrac{3}{2}, \tfrac{3}{2} | L_+ | \tfrac{3}{2}, \tfrac{1}{2} \rangle = \hbar\sqrt{\tfrac{3}{2}\left(\tfrac{3}{2}+1\right) - \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)} = \sqrt{3}\,\hbar$$

$$\langle \tfrac{3}{2}, \tfrac{1}{2} | L_+ | \tfrac{3}{2}, -\tfrac{1}{2} \rangle = \hbar\sqrt{\tfrac{3}{2}\left(\tfrac{3}{2}+1\right) + \tfrac{1}{2}\left(-\tfrac{1}{2}+1\right)} = 2\hbar$$

$$\langle \tfrac{3}{2}, -\tfrac{1}{2} | L_+ | \tfrac{3}{2}, -\tfrac{3}{2} \rangle = \hbar\sqrt{\tfrac{3}{2}\left(\tfrac{3}{2}+1\right) + \tfrac{3}{2}\left(-\tfrac{3}{2}+1\right)} = \sqrt{3}\,\hbar$$

(d) In spherical coordinates a wave function has the form $\psi(r,\theta,\phi) = R(r)cos(\phi)\sin(\theta)$. If a total angular momentum, L^2, measurement is performed on this state, what are the possible outcomes and their associated probabilities?

(L_z also)

		probability	outcomes:
L^2:	$\ell = 1$	100%	$\hbar^2 1(1+1) = 2\hbar^2$
L_z:	$m = +1$	50%	$\hbar$
	-1	50%	$-\hbar$

5. **Radial Schrodinger equation**

A particle moves in the radial potential given by $V(r) = \infty$ for $r < a$ and $V(r) = -V_o$ for $b > r > a$, and $Vr) = 0$ for $r > b$, where we take $V_o > 0$. In the following take the energy to be in the range $-V_o < E < 0$ and ~~take~~ the angular momentum to be zero, $l = 0$.

(a) What is the solution to the $u(r)$ radial equation for $a < r < b$ with $-V_o < E < 0$ and $l = 0$? Eliminate any unphysical solutions.

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} - V_o u = Eu$$

unphysical since $u(a) = 0$

$$\rightarrow u(r) = A\sin(k(r-a)) + \cancel{B\cos(k(r-a))}$$

$$\text{with } k = \sqrt{\frac{2m(E+V_o)}{\hbar^2}}$$

(b) What is the solution to the $u(r)$ radial equation for $r > b$ with $-V_o < E < 0$ and $l = 0$? Eliminate any unphysical solutions as $r \to \infty$.

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = Eu$$

unphysical since $u \to 0$ as $r \to \infty$

$$u(r) = \cancel{A' e^{Kr}} + B' e^{-Kr}$$

$$\text{with } K = \sqrt{\frac{2m(-E)}{\hbar^2}}$$

(c) Derive the boundary conditions for these $u(r)$ at $r = b$

$$u(b) = A\sin(k(b-a)) = B' e^{-Kb}$$

$$u'(b) = Ak\cos(k(b-a)) = -KB' e^{-Kb}$$

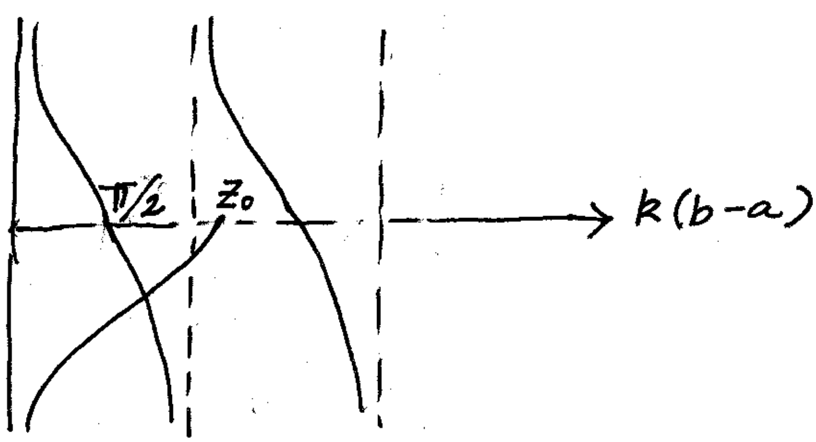
$$\rightarrow \boxed{\tan(k(b-a)) = -\frac{k}{K}}$$

Since $K^2 + k^2 = \frac{2mV_o}{\hbar^2}$, $K = \sqrt{\frac{2mV_o}{\hbar^2} - k^2}$

Let $z_o^2 = \frac{2mV_o}{\hbar^2}(b-a)^2$. Then

$$\cot(k(b-a)) = -\frac{\sqrt{z_o^2 - (k(b-a))^2}}{k(b-a)}$$

(d) Solve the boundary conditions graphically. What is the minimum value of V_o so that there is at least one solution for $-V_o < E < 0$?



In order to have at least one solution,

$$z_o > \frac{\pi}{2} \rightarrow \frac{2mV_o}{\hbar^2}(b-a)^2 > \frac{\pi^2}{4} \rightarrow \boxed{V_o > \frac{\pi^2\hbar^2}{8m(b-a)^2}}$$