Name: Solution

Final Exam - PHY 4604 - Fall 2018
December 12, 2018
12:30P-2:30P, NPB 1002

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

Each exam question, (a),…, (d), is worth 5 points, and the entire exam is out of 100 points. Some formulas are given with the relevant question.

1. **Short answer:**

(a) If $\psi(x)$ is a solution to the time independent Schrodinger equation with energy E, what is the corresponding solution to the time dependent Schrodinger equation?

$$\psi(x) e^{-iEt/\hbar}$$

(b) Write down the wave function, $\psi(r_1, r_2)$, for Bosons in single particles states $\phi_a(r)$ and $\phi_b(r)$.

$$\frac{1}{\sqrt{2}}\left(\varphi_a(r_1)\varphi_b(r_2) + \varphi_b(r_1)\varphi_a(r_2)\right)$$

(c) The orbital configuration of Boron is $1s^2 2s^2 2p$. What is the orbital angular momentum (l) of a Boron atom?

$$l = 1$$

(d) What is the condition that three states, $|\psi_a\rangle$, $|\psi_b\rangle$, and $|\psi_c\rangle$ are orthonormal and are complete?

Orthonormal: $\langle\psi_a|\psi_b\rangle = 0 \quad \langle\psi_a|\psi_a\rangle = 1$

$\langle\psi_a|\psi_c\rangle = 0 \quad \langle\psi_b|\psi_b\rangle = 1$

$\langle\psi_b|\psi_c\rangle = 0 \quad \langle\psi_c|\psi_c\rangle = 1$

Complete: $|\psi_a\rangle\langle\psi_a| + |\psi_b\rangle\langle\psi_b| + |\psi_c\rangle\langle\psi_c| = \mathbb{1}$

2. **One Dimensional Schrodinger Equation:**
Consider the potential $V(x) = V_o > 0$ for $0 < x < a$ and $V(x) = 0$ elsewhere. In the following take the energy to be larger than V_o: $E > V_o > 0$.

(a) What is the general form of the solutions to the time independent Schrodinger equation for $x < 0$, for $0 < x < a$ and for $x > a$? Make sure to define all wave vectors in terms of the energy, E, and V_o.

$x < 0$: $\quad Ae^{ikx} + Be^{-ikx} \qquad k = \sqrt{\frac{2m}{\hbar^2} E}$

$0 < x < a$: $\quad Ce^{ik'x} + De^{-ik'x} \qquad k' = \sqrt{\frac{2m}{\hbar} (E - V_o)}$

$x > a$: $\quad Ee^{ikx} + Fe^{-ikx}$

(b) For a wave incident from the right, which of the coefficients in part (a) is zero?

$A = 0$

(c) What are the boundary conditions at $x = 0$ and at $x = a$ in terms of the coefficients that you defined in part (a) for the case of a wave incident from the right.

$$\cancel{A} + B = C + D$$
$$ik(\cancel{A} - B) = ik'(C - D)$$
$$Ce^{ik'a} + De^{-ik'a} = Ee^{ika} + Fe^{-ika}$$
$$ik'(Ce^{ik'a} - De^{-ik'a}) = ik(Ee^{ika} - Fe^{-ika})$$

(d) Solve for the transmission and reflection probabilities for the special case

$$a\sqrt{\frac{2m}{\hbar^2}(E - V_o)} = \pi \rightarrow k'a = \pi$$
$$a\sqrt{\frac{2m}{\hbar^2}E} = 2\pi. \rightarrow ka = 2\pi$$

Hint: if you substitute for this special case first, the algebra is much easier.

$$\left.\begin{aligned} B &= C + D \\ k(-B) &= k'(C - D) \\ -(C + D) &= E + F \\ -k'(C - D) &= k(E - F) \end{aligned}\right\}$$

$$B = -(E + F) = -E - F$$
$$B = E - F$$

$$\rightarrow E = 0$$
$$B = F$$

$$\rightarrow T = \frac{|B|^2}{|F|^2} = 1 \quad \& \quad R = \frac{|E|^2}{|F|^2} = 0$$

3. **Harmonic Oscillator:**
The Hamiltonian of the one-dimensional harmonic oscillator is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2.$$

Suppose state of system at $t = 0$ is

$$\psi(x,0) = C\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\psi_n(x) = C\left(\psi_0(x) + \frac{\alpha}{\sqrt{1!}}\psi_1(x) + \frac{\alpha^2}{\sqrt{2!}}\psi_2(x) + \cdots\right),$$

where $\psi_n(x)$ for $n = 0, 1, 2, \ldots$ are the energy eigenstates of the Hamiltonian. α is a complex number. Do not be intimidated by the infinite sum. All the things you have learned for finite sums apply here as well.

(a) What is the normalization constant, C? You can simplify your final result using the Taylor series expansion for an exponential:

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

$$1 = \int |\psi(x,0)|^2 dx = |C|^2\left(1 + \frac{|\alpha|^2}{1!} + \frac{|\alpha|^4}{2!} + \cdots\right)$$

$$= |C|^2 e^{|\alpha|^2}$$

$$\rightarrow |C|^2 = e^{-|\alpha|^2} \text{ and } C = e^{-|\alpha|^2/2}$$

(b) What is the wave function, $\psi(x,t)$, at time $t > 0$?

$$\psi(x,t) = C\left(e^{-i\frac{\omega}{2}t}\psi_0(x) + e^{-i\frac{3\omega}{2}t}\frac{\alpha}{\sqrt{1!}}\psi_1(x) + e^{-i\frac{5\omega}{2}t}\frac{\alpha^2}{\sqrt{2!}}\psi_2(x) + \cdots\right)$$

$$= C e^{-i\frac{\omega}{2}t}\sum_{n=0}^{\infty}\frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}\psi_n(x)$$

(c) For two energy eigenstates of the harmonic oscillator, $\psi_n(x)$ and $\psi_k(x)$, what is the matrix element for the position:

$$\langle k|x|n\rangle = \int dx \psi_k^*(x)\sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)\psi_n(x)?$$

How are k and n related for the non-zero matrix elements?

$$\langle k|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{n} \quad \text{if } k=n-1$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\sqrt{n+1} \quad \text{if } k=n+1$$

$$= 0 \quad \text{otherwise}$$

(d) Compute the expectation values of x for the wave function in part (b).

$$\sqrt{\frac{2m\omega}{\hbar}}\langle x\rangle = |C|^2\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha^* e^{i\omega t})^k}{\sqrt{k!}}\frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}\langle k|x|n\rangle$$

$$= |C|^2\sum_{n=0}^{\infty}\left\{\frac{(\alpha^* e^{i\omega t})^{n-1}}{\sqrt{(n-1)!}}\frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}\cdot\sqrt{n} + \frac{(\alpha^* e^{i\omega t})^{n+1}}{\sqrt{(n+1)!}}\frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}\sqrt{n+1}\right\}$$

$$= |C|^2\sum_{n=1}^{\infty}\alpha e^{-i\omega t}\frac{(|\alpha e^{-i\omega t}|^2)^{n-1}}{(n-1)!} + |C|^2\sum_{n=0}^{\infty}\alpha^* e^{i\omega t}\frac{(|\alpha e^{-i\omega t}|^2)^n}{n!}$$

$$\rightarrow \langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\alpha e^{-i\omega t} + \alpha^* e^{i\omega t}\right)$$

4. **Formalism:**
In the following use the Hamiltonian

$$H = E_o \begin{pmatrix} 0 & 0 & i \\ 0 & 2 & 0 \\ -i & 0 & 0 \end{pmatrix}.$$

(a) What are the eigenvalues and eigenvectors of this Hamiltonian?

$2E_o$: $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

E_o: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}$

$-E_o$: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$

(b) The wave function at $t = 0$ is

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

An energy measurement is made at $t = 0$. What are the possible outcomes and the associated probabilities?

outcomes: probabilities:

$2E_o$ $\quad \left|(0\ 1\ 0)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$

E_o $\quad \left|\frac{1}{\sqrt{2}}(1\ 0\ i)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right|^2 = \left|\frac{1}{2}\right|^2 = \frac{1}{4}$

$-E_o$ $\quad \left|\frac{1}{\sqrt{2}}(1\ 0\ -i)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right|^2 = \left|\frac{1}{2}\right|^2 = \frac{1}{4}$

(c) If instead of making an energy measurement at $t = 0$, the system is allowed to evolve in time, what is the wave function at time t:

$$|\psi(t)\rangle = \begin{pmatrix} a(t) \\ b(t) \\ c(t) \end{pmatrix}?$$

Give explicit expressions for $a(t)$, $b(t)$, and $c(t)$.

$$|\psi(t)\rangle = e^{-i2E_0t/\hbar}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{2\sqrt{2}}e^{-iE_0t/\hbar}\begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} + \frac{1}{2\sqrt{2}}e^{iE_0t/\hbar}\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$$

$$a(t) = \frac{1}{\sqrt{2}}\cos(E_0t/\hbar)$$

$$b(t) = \frac{1}{\sqrt{2}}e^{-i2E_0t/\hbar}$$

$$c(t) = \frac{-1}{\sqrt{2}}\sin(E_0t/\hbar)$$

(d) For this hamiltonian find a state for which an energy measurement yields $+E_o$ 50% of the time, $2E_o$ 25% of the time, and $-E_0$ 25% of the time. (There are many right answers to this problem.)

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} = \boxed{\begin{pmatrix} \frac{1}{2} + \frac{1}{2\sqrt{2}} \\ 1/2 \\ \left(\frac{1}{2\sqrt{2}} - \frac{1}{2}\right)i \end{pmatrix}}$$

check normalization:

$$\left|\frac{1}{2} + \frac{1}{2\sqrt{2}}\right|^2 + \left|\frac{1}{2}\right|^2 + \left|\left(\frac{1}{2\sqrt{2}} - \frac{1}{2}\right)i\right|^2$$

$$= 2\left(\frac{1}{4} + \frac{1}{8}\right) + \frac{1}{4} = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} \stackrel{\checkmark}{=} 1$$

5. **Angular momentum:**

(a) Compute the commutator $[L_z, L_+L_-]$=

$$= [L_z, L_+]L_- + L_+[L_z, L_-]$$

$$= i\hbar\Big(\underbrace{(L_y - iL_x)}_{-iL_+}L_- + L_+\underbrace{(L_y + iL_x)}_{iL_-}\Big)$$

$$= \hbar(L_+L_- - L_+L_-) = 0$$

(b) What is the matrix element $\langle\frac{3}{2},\frac{3}{2}|J_xJ_z|\frac{3}{2},\frac{1}{2}\rangle$?

$$\langle\tfrac{3}{2},\tfrac{3}{2}|J_xJ_z|\tfrac{3}{2},\tfrac{1}{2}\rangle = \frac{\hbar}{2}\langle\tfrac{3}{2},\tfrac{3}{2}|L_x|\tfrac{3}{2},\tfrac{1}{2}\rangle$$

$$= \frac{\hbar}{2}\langle\tfrac{3}{2},\tfrac{3}{2}|\frac{L_+}{2}|\tfrac{3}{2},\tfrac{1}{2}\rangle$$

$$= \frac{\hbar}{4}\hbar\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)}$$

$$= \frac{\hbar^2}{4}\sqrt{\frac{15-3}{4}} = \frac{\sqrt{3}}{4}\hbar^2$$

(c) In adding $j_1 = 2$ and $j_2 = 1$ angular momentum, the state with total angular momentum $j = 3$ and $m = 2$ is

$$|j = 3, m = 2\rangle = \sqrt{\frac{1}{3}}|2, 2\rangle \otimes |1, 0\rangle + \sqrt{\frac{2}{3}}|2, 1\rangle \otimes |1, 1\rangle.$$

What is the $|j = 2, m = 2\rangle$ state?

$$\sqrt{\frac{2}{3}}\,|2,2\rangle \otimes |1,0\rangle - \sqrt{\frac{1}{3}}\,|2,1\rangle \otimes |1,1\rangle$$

(d) If ~~the~~ J_z for particle one with $j_1 = 2$ is measured for the $|j = 3, m = 2\rangle$ state, what are the possible outcomes and their associated probabilities?

$J_z^{(1)}$	Probability
$2\hbar$	$1/3$
$\hbar$	$2/3$