Gaussian wavepacket:

$$\varphi(k) = C e^{-\alpha(k-k_0)^2}.$$

An integral: $\mathcal{I} = \int_{-\infty}^{+\infty} dx\, e^{-\alpha x^2}$

$$\mathcal{I}^2 = \int dx \int dy\, e^{-\alpha(x^2+y^2)}$$

$$= \int_0^\infty 2\pi r\, dr\, e^{-\alpha r^2}$$

$$= \int_0^\infty \pi\, d(r^2)\, e^{-\alpha r^2}$$

$$= \pi/\alpha$$

$\Rightarrow \mathcal{I} = \sqrt{\frac{\pi}{\alpha}}$. //

Now, $|\varphi(k)|^2 = C^2 e^{-2\alpha(k-k_0)^2}$ and normalization cond. is

$$1 = \int dk\, |\varphi(k)|^2 = C^2 \int dk\, e^{-2\alpha(k-k_0)^2} = C^2 \frac{\sqrt{\pi}}{\sqrt{2\alpha}}$$

$\rightarrow C = \left(\frac{2\alpha}{\pi}\right)^{1/4}$

in realspace:

$$\psi(x,0) = \int \frac{dk}{\sqrt{2\pi}}\, C e^{-\alpha(k-k_0)^2} e^{ikx}$$

$$= \int \frac{dk}{\sqrt{2\pi}}\, C\, e^{-\alpha(k-k_0)^2} e^{i(k-k_0)x}\, e^{ik_0x}$$

Complete the square:

$$\alpha(k-k_0)^2 - i(k-k_0)x = \alpha\left\{(k-k_0)^2 - \frac{ix}{\alpha}(k-k_0)\right\}$$

$$= \alpha\left\{(k-k_0)^2 - \frac{ix}{\alpha}(k-k_0) - \frac{x^2}{4\alpha^2} + \frac{x^2}{4\alpha^2}\right\}$$

$$= \alpha\left(k-k_0-\frac{ix}{2\alpha}\right)^2 + \frac{x^2}{4\alpha}$$

$$\Rightarrow \psi(x,0) = e^{ik_0x}\, e^{-\frac{x^2}{4\alpha}} \int \frac{dk}{\sqrt{2\pi}}\, C\, e^{-\alpha\left(k-k_0-\frac{ix}{2\alpha}\right)^2}$$

$$= \frac{C}{\sqrt{2\pi}}\sqrt{\frac{\pi}{\alpha}}\, e^{ik_0x}\, e^{-\frac{x^2}{4\alpha}}$$, since $e^{-\alpha z^2}$ has no poles we can shift the integral up in the complex plane.

$$= \frac{(2\alpha/\pi)^{1/4}}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{\sqrt{\alpha}}\, e^{ik_0x}\, e^{-\frac{x^2}{4\alpha}}$$

$$= \frac{1}{(2\pi\alpha)^{1/4}}\, e^{ik_0x}\, e^{-\frac{x^2}{4\alpha}}$$

check: $\int dx\, |\psi(x,0)|^2 = \frac{1}{\sqrt{2\pi\alpha}} \int dx\, e^{-\frac{x^2}{2\alpha}}$

$$= \frac{1}{\sqrt{2\pi\alpha}} \frac{\sqrt{\pi}}{\sqrt{1/2\alpha}} = 1 \checkmark$$

time evolution:

$$\psi(x,t) = \int \frac{dk}{\sqrt{2\pi}} \varphi(k) e^{ikx} e^{-i\omega_k t}$$

$$= \int \frac{dk}{\sqrt{2\pi}} C e^{-\alpha(k-k_0)^2} e^{ikx} e^{-i\frac{\hbar k^2}{2m}t}$$

Again complete the square:

$$\alpha(k-k_0)^2 - ikx + i\frac{\hbar k^2}{2m}t =$$

$$= \left(\alpha + \frac{i\hbar}{2m}t\right)k^2 + (-2\alpha k_0 - ix)k + \alpha k_0^2$$

$$= \left(\alpha + \frac{i\hbar}{2m}t\right)\left\{k^2 + \frac{(-2\alpha k_0 - ix)}{\left(\alpha + \frac{i\hbar}{2m}t\right)}k + \frac{(-2\alpha k_0 - ix)^2}{4\left(\alpha + \frac{i\hbar}{2m}t\right)^2}\right\}$$

$$- \frac{(-2\alpha k_0 - ix)^2}{4\left(\alpha + \frac{i\hbar}{2m}t\right)^2} + \alpha k_0^2$$

$$\Rightarrow \psi(x,t) = \frac{C}{\sqrt{2\pi}} \exp\left(\frac{(-2\alpha k_0 - ix)^2}{4\left(\alpha + \frac{i\hbar}{2m}t\right)} - \alpha k_0^2\right) \times$$

$$\times \int dk \, \exp\left\{-\left(\alpha + \frac{i\hbar}{2m}t\right)\left(k - \frac{(-2\alpha k_0 - ix)}{2\left(\alpha + \frac{i\hbar}{2m}t\right)}\right)^2\right\}$$

The integral, $\int dx \, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$, is still valid for α complex using the same trick as before. The square root must have a positive real part.

$$\Rightarrow \psi(x,t) = \frac{C}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{\sqrt{\alpha + \frac{i\hbar}{2m}t}} \exp\left(\frac{(-2\alpha k_0 - ix)^2}{4(\alpha + \frac{i\hbar}{2m}t)} - \alpha k_0^2 \right)$$

The exponent is:

cancel

$$\frac{(-2\alpha k_0 - ix)^2}{4(\alpha + \frac{i\hbar}{2m}t)} - \alpha k_0^2 = \frac{-x^2 + 4i\alpha k_0 x + 4\alpha^2 k_0^2 - \alpha k_0^2 (4\alpha + \frac{2i\hbar}{m}t)}{4(\alpha + \frac{i\hbar}{2m}t)}$$

$$= \frac{-x^2 + 4i\alpha k_0 x - 2\alpha k_0^2 \frac{i\hbar}{m} t}{4(\alpha + \frac{i\hbar}{2m}t)}$$

Compute the real and imaginary parts of the exponent.

$$\mathrm{Re}\left\{ \frac{-x^2}{4(\alpha + \frac{i\hbar}{2m}t)} \right\} = \frac{-\alpha x^2}{4(\alpha^2 + (\frac{\hbar t}{2m})^2)}$$

$$\mathrm{Re}\left\{ \frac{4i\alpha k_0 x}{4(\alpha + \frac{i\hbar}{2m}t)} \right\} = \frac{+4\alpha k_0 x \frac{\hbar t}{2m}}{4(\alpha^2 + (\frac{\hbar t}{2m})^2)}$$

$$\mathrm{Re}\left\{ \frac{-2\alpha k_0^2 \frac{i\hbar}{m}t}{4(\alpha + \frac{i\hbar}{2m}t)} \right\} = \frac{-2\alpha k_0^2 \frac{\hbar}{m}t \frac{\hbar}{2m}t}{4(\alpha^2 + (\frac{\hbar t}{2m})^2)}$$

$$\rightarrow \mathrm{Re}\{\text{exponent}\} = \frac{-\alpha}{4(\alpha^2 + (\frac{\hbar t}{2m})^2)} \left(x^2 - 2\frac{\hbar k_0}{m} t x + (\frac{\hbar k_0}{m} t)^2 \right)$$

$$= \frac{-\alpha}{4(\alpha^2 + (\frac{\hbar t}{2m})^2)} \left(x - \underbrace{\frac{\hbar k_0}{m}}_{\text{velocity}} t \right)^2$$

$$\text{Im}\{\text{exponent}\} = \frac{x^2 \frac{\hbar t}{2m} + 4\alpha^2 k_0 x - 2\alpha^2 k_0^2 \frac{\hbar t}{m}}{4\left(\alpha^2 + \left(\frac{\hbar t}{2m}\right)^2\right)}$$

We expect Im{exponent} to contain $k_0 x - \frac{\hbar k_0^2}{2m} t$ so add and subtract it.

$$\text{Im}\{\text{exponent}\} = k_0 x - \frac{\hbar k_0^2}{2m} t + \frac{x^2 \frac{\hbar t}{2m} - 4\left(\frac{\hbar t}{2m}\right)^2 k_0 x + 4\left(\frac{\hbar t}{2m}\right)^2 \frac{\hbar k_0^2 t}{2m}}{4\left(\alpha^2 + \left(\frac{\hbar t}{2m}\right)^2\right)}$$

$$= k_0 x - \frac{\hbar k_0^2}{2m} t + \frac{\frac{\hbar t}{2m}}{4\left(\alpha^2 + \left(\frac{\hbar t}{2m}\right)^2\right)} \left(x^2 - 2\frac{\hbar k_0 t}{m} x + \left(\frac{\hbar k_0 t}{m}\right)^2\right)$$

$$= k_0 x - \frac{\hbar k_0^2}{2m} t + \frac{\frac{\hbar t}{2m}}{4\left(\alpha^2 + \left(\frac{\hbar t}{2m}\right)^2\right)} \left(x - \frac{\hbar k_0 t}{m}\right)^2$$

$$\Rightarrow \text{exponent} = i\left(k_0 x - \frac{\hbar k_0^2}{2m} t\right) - \frac{\left(x - \frac{\hbar k_0}{m} t\right)^2}{4\left(\alpha + \frac{i\hbar t}{2m}\right)}.$$

Since $C = (2\alpha/\pi)^{1/4}$,

$$\boxed{\psi(x,t) = \frac{1}{(2\pi)^{1/4}} \frac{\alpha^{1/4}}{\sqrt{\alpha + \frac{i\hbar}{2m} t}} e^{i\left(k_0 x - \frac{\hbar k_0^2}{2m} t\right)} \exp\left(\frac{-\left(x - \frac{\hbar k_0}{m} t\right)^2}{4\left(\alpha + \frac{i\hbar t}{2m}\right)}\right),}$$

and $$\boxed{|\psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}} \frac{\alpha^{1/2}}{\sqrt{\alpha^2 + \left(\frac{\hbar t}{2m}\right)^2}} \exp\left(\frac{-\alpha\left(x - \frac{\hbar k_0}{m} t\right)^2}{2\left(\alpha^2 + \left(\frac{\hbar t}{2m}\right)^2\right)}\right).}$$

Change variables so x & t are dimensionless:

$$\boxed{\chi = x/\sqrt{\alpha}} \quad \text{and} \quad \boxed{\tau = \frac{\hbar t}{m\alpha}}.$$

$$\rightarrow \psi(x,t) = \frac{1}{(2\pi)^{1/4}} \frac{1}{\alpha^{1/4}} \frac{1}{\sqrt{1+i\tau/2}} e^{i(k_0\sqrt{\alpha}\,\chi - \frac{1}{2}(k_0\sqrt{\alpha})^2\tau)}$$

$$\times \exp\left(-\frac{(\chi - k_0\sqrt{\alpha}\,\tau)^2}{4(1+i\tau/2)}\right)$$

The group velocity is $\hbar k_0/m$. To make it dimensionless multiply by $\frac{1}{\sqrt{\alpha}}\frac{m\alpha}{\hbar}$, where $[\sqrt{\alpha}] = L$ and $\left[\frac{m\alpha}{\hbar}\right] = T$.

$$\boxed{v = \frac{\hbar k_0}{m}\frac{1}{\sqrt{\alpha}}\frac{m\alpha}{\hbar} = k_0\sqrt{\alpha}}$$

The normalization is

$$\int_{-\infty}^{+\infty} dx\,|\psi(x,t)|^2 = 1 = \int_{-\infty}^{+\infty} \frac{d\chi}{\sqrt{\alpha}}\,\sqrt{\alpha}\,|\psi(x,t)|^2 = \int_{-\infty}^{+\infty} d\chi\,|\psi(\chi,\tau)|^2,$$

where $\psi(\chi,\tau) = (\alpha)^{1/4}\,\psi(x,t)$.

$$\boxed{\psi(\chi,\tau) = \frac{1}{(2\pi)^{1/4}} \frac{1}{\sqrt{1+i\tau/2}} e^{i(v(\chi - \frac{1}{2}v\tau))} e^{-\frac{(\chi - v\tau)^2}{4(1+i\tau/2)}}}$$