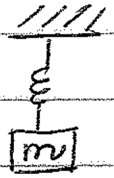
Harmonic oscillator:



$$ma = m\frac{d^2x}{dt^2} = -kx$$

$$\rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad \underbrace{\omega^2 = \frac{k}{m}}_{k = m\omega^2}, \quad \omega = \sqrt{\frac{k}{m}}$$

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$\text{Energy} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$= \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2x^2$$

$$= \frac{1}{2m}\left[p^2 + (m\omega x)^2\right]$$

For the Schrodinger equation

$$p = \frac{\hbar}{i}\frac{d}{dx} \text{ and } \quad \frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 + (m\omega x)^2\right]\psi = E\psi$$

$$\text{or } \frac{1}{2m}\left[-\hbar^2\frac{d^2}{dx^2} + (m\omega x)^2\right]\psi = E\psi$$

$$\text{or } \boxed{\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2\right]\psi = E\psi}$$

We will solve this differential equation by two methods.

Algebraic or Operator Method:

Consider: $$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x)$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)$$

$$a_- = \frac{1}{\sqrt{2\hbar m\omega}}(ip + m\omega x)$$

$$= \frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{d}{dx} + m\omega x\right)$$

Why?

$$a_+ a_- f(x) = \frac{1}{2\hbar m\omega}\left(-\hbar\frac{d}{dx} + m\omega x\right)\left(\hbar\frac{d}{dx} + m\omega x\right)f$$

$$= \frac{1}{2\hbar m\omega}\left(-\hbar^2\frac{d^2f}{dx^2} + m^2\omega^2x^2 f \underbrace{- \hbar\frac{d}{dx}m\omega x f + m\omega x\hbar\frac{d}{dx}f}_{-\hbar m\omega f}\right)$$

$$\rightarrow \hbar\omega a_+ a_- f = \frac{1}{2m}\left(-\hbar^2\frac{d^2f}{dx^2} + m^2\omega^2x^2 f\right) - \frac{\hbar\omega}{2}f$$

$$\rightarrow \boxed{Hf = \hbar\omega\left(a_+ a_- + \frac{1}{2}\right)f}$$

Commutator: (s)

$$(xp - px)f = [x, p]f$$

$$= \left[x \frac{\hbar}{i}\frac{df}{dx} - \frac{\hbar}{i}\frac{d(xf)}{dx}\right]$$

$$= -\frac{\hbar}{i} f = i\hbar f$$

$$\Rightarrow \boxed{[x, p] = i\hbar}$$

$$[A, (B+C)] = A(B+C) - (B+C)A$$

$$= AB - BA + AC - CA$$

$$= [A, B] + [A, C]$$

Similarly

$$[B+C, A] = [B, A] + [C, A]$$

Also

$$[A, B] = -[B, A]$$

$$[A, A] = AA - AA = 0$$

$$[\alpha A, B] = \alpha [A, B]$$

A, B, C operators
α = number

Apply these:

$$[a_+, a_-] = \frac{1}{2\hbar m\omega}[-ip + m\omega x, ip + m\omega x]$$

$$= \frac{1}{2\hbar m\omega}\Big\{ \underbrace{\cancel{[-ip, ip]}}_{0} + \underbrace{\cancel{[m\omega x, m\omega x]}}_{0} + \underbrace{[-ip, m\omega x]}_{-im\omega(-i\hbar)} + \underbrace{[m\omega x, ip]}_{im\omega(i\hbar)} \Big\}$$

$\rightarrow [a_+, a_-] = -1$

or $\boxed{[a_-, a_+] = 1}$ $\qquad a_-a_+ = a_+a_- + 1$

Implications: $\quad H\psi = E\psi$

$$H(a_+\psi) = \hbar\omega\left(a_+a_- + \frac{1}{2}\right)a_+\psi$$

$$= \hbar\omega\left(a_+a_-a_+ + \frac{1}{2}a_+\right)\psi$$

$$= a_+\hbar\omega\left(a_-a_+ + \frac{1}{2}\right)\psi$$

$$= a_+\hbar\omega\left(a_+a_- + \frac{1}{2} + 1\right)\psi$$

$$= (E + \hbar\omega)(a_+\psi)$$

Raising operator raises the energy by $\hbar\omega$.
Similarly,

$$H(a_-\psi) = (E - \hbar\omega)(a_-\psi)$$

Lowering operator reduces energy by $\hbar\omega$.

For any ψ

$$\langle E \rangle = \int \psi^* \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 \right) \psi$$

$$= \int \frac{\hbar^2}{2m}\left(\frac{d\psi^*}{dx}\right)\left(\frac{d\psi}{dx}\right) dx \leftarrow \text{after integration by parts} \qquad \geq 0$$

$$+ \frac{1}{2}m\omega^2 \int (x\psi^*)(x\psi)\, dx \qquad \geq 0$$

$$\geq 0$$

Thus, the eigenvalues of H are bounded from below (by 0).

Since a_- lowers the energy by $-\hbar\omega$, there must be some state ψ_0 for which

$$\boxed{a_- \psi_0 = 0.} \leftarrow \text{ground state}$$

Otherwise the energy would decrease as out bound as one continued to apply a_-. ψ_0 has energy $\hbar\omega/2$:

$$\boxed{H\psi_0 = \hbar\omega\left(a_+ a_- + \frac{1}{2}\right)\psi_0 = (\hbar\omega/2)\,\psi_0}$$

Other eigenstates are related via

$$\boxed{H(a_+)^n \psi_0 = (\hbar\omega)\left(n+\frac{1}{2}\right)(a_+)^n \psi_0.}$$