Hermitian operators: $\boxed{Q = Q^\dagger}$

Position (x), momentum, and the Hamiltonian are all Hermitian operators.

The eigenvalues of Hermitian operators are real:

$$Q|\psi\rangle = \lambda|\psi\rangle$$

eigenvector eigenvalue

$$\rightarrow \langle\psi|Q\psi\rangle = \lambda\langle\psi|\psi\rangle$$

$$= \langle Q\psi|\psi\rangle = \lambda^*\langle\psi|\psi\rangle$$

$\rightarrow \lambda = \lambda^*$, ie. λ is real

If Q is Hermitian, $Q|\psi_1\rangle = \lambda_1|\psi_1\rangle$,
$Q|\psi_2\rangle = \lambda_2|\psi_2\rangle$, and $\lambda_1 \neq \lambda_2$,
then $\langle\psi_1|\psi_2\rangle = 0$.

States are orthogonal with different eigenvalues of Hermitian operators.

Proof:

$$\langle\psi_2|Q\psi_1\rangle = \langle Q\psi_2|\psi_1\rangle$$
$$= \lambda_1\langle\psi_2|\psi_1\rangle = \lambda_2\langle\psi_2|\psi_1\rangle$$

$$\rightarrow (\lambda_1 - \lambda_2)\langle\psi_2|\psi_1\rangle = 0$$
$$\rightarrow \langle\psi_2|\psi_1\rangle = 0 \text{ since } \lambda_1 \neq \lambda_2.$$

(Even when $\lambda_1 = \lambda_2$, there is a procedure to create orthogonal eigenstates.)

Projection operators:

Let $Q|\psi_n\rangle = \lambda_n|\psi_n\rangle$

Then $Q|\psi\rangle = Q\sum_n c_n|\psi_n\rangle$

$$= \sum_n \lambda_n c_n|\psi_n\rangle,$$

where $c_n = \langle\psi_n|\psi\rangle$.

$$\rightarrow Q|\psi\rangle = \sum_n \lambda_n|\psi_n\rangle\langle\psi_n|\psi\rangle$$

$$= \Big(\underbrace{\sum_n \lambda_n|\psi_n\rangle\langle\psi_n|}_{\text{operator}}\Big)|\psi\rangle$$

$$\boxed{Q = \sum_n \lambda_n|\psi_n\rangle\langle\psi_n|}$$

The $|\psi_n\rangle\langle\psi_n|$ are projection operators.

$$(|\psi_n\rangle\langle\psi_n|)|\psi\rangle = c_n|\psi_n\rangle.$$

Projecting onto each $|\psi_n\rangle$ and adding the result, $c_n|\psi_n\rangle$, gives back $|\psi\rangle$:

$$\boxed{\mathbb{1} = \sum_n |\psi_n\rangle\langle\psi_n|}$$

Example: $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\det(Q - \lambda \mathbb{1}) = \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0$$

$$\rightarrow \lambda = \pm 1$$

$\lambda_1 = 1$ $\qquad\qquad$ $\lambda_2 = -1$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$\rightarrow c_1 = c_2$ $\qquad\qquad$ $\rightarrow c_1 = -c_2$

$\rightarrow |\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad$ $|\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\langle\psi_1| = \frac{1}{\sqrt{2}}(1 \;\; 1)$ $\qquad$ $\langle\psi_2| = \frac{1}{\sqrt{2}}(1 \; -1)$

$$|\psi_1\rangle\langle\psi_1| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}(1 \;\; 1) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$|\psi_2\rangle\langle\psi_2| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\frac{1}{\sqrt{2}}(1 \; -1) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

check: $|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| = \mathbb{1}$ ✓

$\lambda_1 |\psi_1\rangle\langle\psi_1| + \lambda_2 |\psi_2\rangle\langle\psi_2| = Q$ ✓