Linear operators:

Operator: vector $\rightarrow$ vector

Linear operator, Q, satisfies

$$Q(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) = \alpha\, Q|\psi_1\rangle + \beta\, Q|\psi_2\rangle.$$

Position, momentum, and the Hamiltonian are all linear operators.

$$x(\psi_1 + \psi_2) = x\psi_1 + x\psi_2$$

$$p(\alpha\psi_1 + \beta\psi_2) = \frac{\hbar}{i}\frac{d}{dx}(\alpha\psi_1 + \beta\psi_2)$$

$$= \alpha\frac{\hbar}{i}\frac{d\psi_1}{dx} + \beta\frac{\hbar}{i}\frac{d\psi_2}{dx}$$

$$= \alpha \cdot p\psi_1 + \beta p\psi_2$$

Let $|\psi'\rangle = Q|\psi\rangle$, where Q is a linear op.

Expand in an orthonormal basis:

$$|\psi\rangle = \sum_n c_n |\psi_n\rangle$$

$$|\psi'\rangle = \sum_n c_n' |\psi_n\rangle$$

$$\rightarrow \quad |\psi'\rangle = \sum_n c_n' |\psi_n\rangle = Q \sum_n c_n |\psi_n\rangle$$

linearity

$$= \sum_n c_n Q |\psi_n\rangle$$

Take the inner product with $|\psi_m\rangle$:

$$\sum_n c_n' \underbrace{\langle \psi_m | \psi_n \rangle}_{\delta_{nm}} = \sum_n c_n \langle \psi_m | Q | \psi_n \rangle$$

$$\Rightarrow \boxed{c_m' = \sum_n \langle \psi_m | Q | \psi_n \rangle c_n}$$

This is <u>matrix</u> <u>multiplication</u>:

$$\begin{pmatrix} c_1' \\ c_2' \\ \vdots \end{pmatrix} = \begin{pmatrix} \langle \psi_1 | Q | \psi_1 \rangle & \langle \psi_1 | Q | \psi_2 \rangle & \cdots \\ \langle \psi_2 | Q | \psi_1 \rangle & \langle \psi_2 | Q | \psi_2 \rangle & \cdots \\ \langle \psi_3 | Q | \psi_1 \rangle & \langle \psi_3 | Q | \psi_2 \rangle & \cdots \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$$

Adjoint: $Q^\dagger$ satisfies:

$$\langle f | Q g \rangle = \langle Q^\dagger f | g \rangle$$

Examples:

$$\langle f | x g \rangle = \int dx \, f^*(x) \, x \, g(x)$$

$$= \int dx \, (x f(x))^* g(x)$$

$$= \langle x f | g \rangle \Rightarrow x^\dagger = x$$

$$\langle f | \frac{dg}{dx} \rangle = \int dx \, f^*(x) \frac{dg}{dx}$$

integration by parts

$$= -\int dx \, \frac{df^*}{dx} g(x)$$

$$= \langle -\frac{df}{dx} | g \rangle \Rightarrow \left(\frac{d}{dx}\right)^\dagger = -\frac{d}{dx}$$

$$\langle f | p g \rangle = \int dx \, f^*(x) \frac{\hbar}{i} \frac{dg}{dx}$$

integration by parts

$$= -\int dx \, \frac{\hbar}{i} \frac{df^*}{dx} g(x)$$

$$= \int dx \left(\frac{\hbar}{i} \frac{df}{dx} \right)^* g(x)$$

$$= \langle p f | g \rangle \Rightarrow p^\dagger = p$$

Adjoint in matrix form:

$$\langle f | Qg \rangle = \langle Q^\dagger f | g \rangle = \langle g | Q^\dagger f \rangle^*$$

$$\rightarrow \langle g | Q^\dagger f \rangle = \langle f | Qg \rangle^*$$

Also written as

$$\langle g | Q^\dagger | f \rangle = \langle f | Q | g \rangle^*.$$

Since this is true for any $|f\rangle$ and $|g\rangle$, let $|g\rangle = |\psi_m\rangle$ and $|f\rangle = |\psi_n\rangle$.

$$\langle \psi_m | Q^\dagger | \psi_n \rangle = \langle \psi_n | Q | \psi_m \rangle^*$$

$Q^\dagger$ is the complex conjugate transpose of Q: $Q^\dagger = (Q^t)^*$. This is also called the Hermitian conjugate.

Example: $\begin{pmatrix} 1 & 2i \\ +3i & 4 \end{pmatrix}^\dagger = \begin{pmatrix} 1 & -3i \\ -2i & 4 \end{pmatrix}$