

N-Particle Systems:

So as not to be too abstract we take $N=2$.
 $|\psi(\vec{r}_1, \vec{r}_2)|^2$ is now the probability density of finding particle 1 at \vec{r}_1 and particle 2 at \vec{r}_2 .

$$\int d^3r_1 \int d^3r_2 |\psi(\vec{r}_1, \vec{r}_2)|^2 = 1$$

If we allow the particles to have spin, then

$$\sum_{s_1} \sum_{s_2} \int d^3r_1 \int d^3r_2 |\psi(\vec{r}_1, s_1, \vec{r}_2, s_2)|^2 = 1.$$

The Schrodinger equation is $i\hbar \frac{\partial \psi}{\partial t} = H \psi$.

$$\begin{aligned} \text{We take } H &= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(\vec{r}_1) + V(\vec{r}_2). \\ &= -\frac{\hbar^2 \nabla_1^2}{2m} - \frac{\hbar^2 \nabla_2^2}{2m} + V(\vec{r}_1) + V(\vec{r}_2). \end{aligned}$$

The particles are taken to be identical.

For identical particles it doesn't make sense to discuss which particle is at \vec{r}_1 & which is at \vec{r}_2 because the particles are indistinguishable.

Mathematically this is expressed using the permutation operator, P .

$$P\psi(1, 2) = \psi(2, 1), \text{ where "1" } = (\vec{r}_1, s_1) \\ \text{and "2" } = (\vec{r}_2, s_2).$$

$$P^2\psi(1, 2) = P\psi(2, 1) = \psi(1, 2)$$

$$\Rightarrow P^2 = 1$$

Thus, P has eigenvalues ± 1 .

$$H\psi = \left(-\frac{\hbar^2 \nabla_1^2}{2m} - \frac{\hbar^2 \nabla_2^2}{2m} + V(r_1) + V(r_2) \right) \psi(1, 2)$$

$$PH\psi = \left(-\frac{\hbar^2 \nabla_2^2}{2m} - \frac{\hbar^2 \nabla_1^2}{2m} + V(r_2) + V(r_1) \right) \psi(2, 1)$$

$$P\psi = \psi(2, 1)$$

$$HP\psi = \left(-\frac{\hbar^2 \nabla_2^2}{2m} - \frac{\hbar^2 \nabla_1^2}{2m} + V(r_2) + V(r_1) \right) \psi(2, 1)$$

$$\text{Thus, } PH\psi = HP\psi \rightarrow \boxed{[P, H]\psi = 0}$$

We can find simultaneous eigenvectors of H and P .

$$P\psi = +1\psi \quad (\text{Boson})$$

$$P\psi = -1\psi \quad (\text{Fermion})$$

$$\text{Suppose } \left(-\frac{\hbar^2 \nabla^2}{2m} + V \right) \varphi_a = E_a \varphi_a$$

$$\left(-\frac{\hbar^2 \nabla^2}{2m} + V \right) \varphi_b = E_b \varphi_b$$

Then

$$\begin{aligned} \left(-\frac{\hbar^2 \nabla_1^2}{2m} - \frac{\hbar^2 \nabla_2^2}{2m} + V(r_1) + V(r_2) \right) \varphi_a(1) \varphi_b(2) \\ = (E_a + E_b) \varphi_a(1) \varphi_b(2) \end{aligned}$$

$$\begin{aligned} \text{and } \left(-\frac{\hbar^2 \nabla_1^2}{2m} - \frac{\hbar^2 \nabla_2^2}{2m} + V(r_1) + V(r_2) \right) \varphi_b(1) \varphi_a(2) \\ = (E_a + E_b) \varphi_b(1) \varphi_a(2) \end{aligned}$$

→ Both $\varphi_a(1)\varphi_b(2)$ and $\varphi_b(1)\varphi_a(2)$ are solutions to the Schrodinger equation; however,

$$P \varphi_a(1)\varphi_b(2) = \varphi_a(2)\varphi_b(1)$$

$$P \varphi_b(1)\varphi_a(2) = \varphi_a(1)\varphi_b(2)$$

They are not eigenvectors of P .

$$\psi_s = \frac{1}{\sqrt{2}} (\varphi_a(1)\varphi_b(2) + \varphi_b(1)\varphi_a(2)) \dots \text{symmetric}$$

$$\psi_A = \frac{1}{\sqrt{2}} (\varphi_a(1)\varphi_b(2) - \varphi_b(1)\varphi_a(2)) \dots \text{anti-symmetric}$$

$$P \psi_s = +1 \psi_s$$

$$P \psi_A = -1 \psi_A$$

If $a=b$, then $\psi_A = 0$. This is the Pauli exclusion principle that two Fermions can not be in the same state.

For $a=b$, $\psi_s \neq 0$. Indeed the correct normalization is $\psi_s = \varphi_a(1)\varphi_a(2)$. You can have any number of bosons in the same state.

Sometimes ψ_A is written as

$$\psi_A = \frac{1}{\sqrt{2}} \begin{vmatrix} \varphi_a(1) & \varphi_a(2) \\ \varphi_b(1) & \varphi_b(2) \end{vmatrix} \quad (\text{Slater determinant})$$