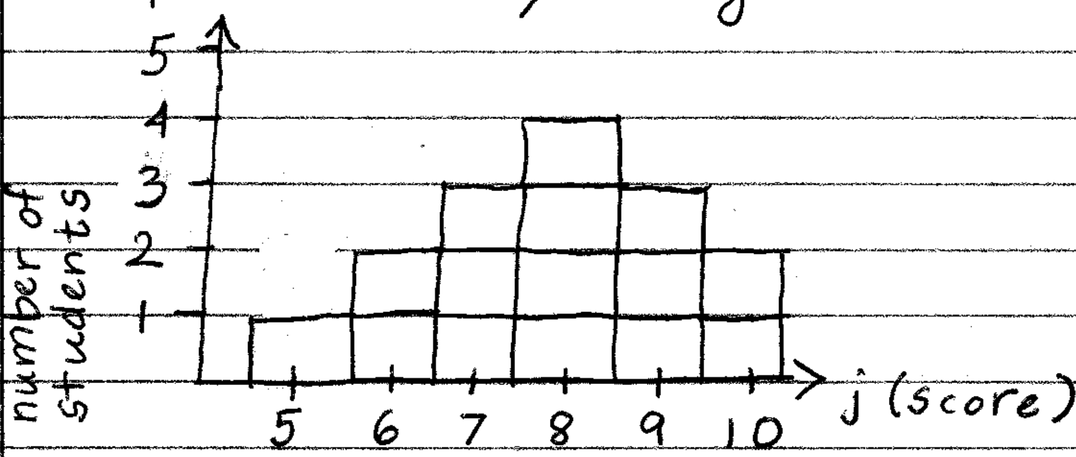
Probability and Normalization:

Example: Probability in a grade distribution



$$\frac{1}{15}, \frac{2}{15}, \frac{3}{15}, \frac{4}{15}, \frac{3}{15}, \frac{2}{15} = P(j) = \text{probability of score}$$

$$\langle j \rangle = \sum_j j\, P(j) = \text{average value}$$

$$\Delta j = j - \langle j \rangle = \text{deviation from average value}$$

$$\langle \Delta j \rangle = \sum_j (j - \langle j \rangle) P(j)$$

$$= \sum_j j\, P(j) - \sum_j \langle j \rangle P(j) \quad \text{number}$$

$$= \langle j \rangle - \langle j \rangle \quad \text{since } \sum_j P(j) = 1$$

$$= 0$$

→ Look at $\langle(\Delta j)^2\rangle =$

$$= \sum_j (j - \langle j \rangle)^2 P(j)$$

$$= \sum_j (j^2 - 2j\langle j \rangle + \langle j \rangle^2) P(j)$$

$$= \langle j^2 \rangle - 2\langle j \rangle\langle j \rangle + \langle j \rangle^2$$

$$= \langle j^2 \rangle - \langle j \rangle^2$$

Usually written as a standard deviation:

$$\sigma_j \equiv \sqrt{\langle(\Delta j)^2\rangle} = \sqrt{\langle j^2 \rangle - \langle j \rangle^2}.$$

Continuous case:

$$\rho(x,t)\,dx = |\psi(x,t)|^2 dx$$

= Probability particle between x & $x+dx$ at time t

probability density

$$\langle x \rangle = \int_{-\infty}^{+\infty} x\,\rho(x,t)\,dx$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 \rho(x,t)\,dx$$

$\vdots$

$$\langle f(x) \rangle = \int_{-\infty}^{+\infty} f(x)\,\rho(x,t)\,dx$$

A particularly useful continuous case is a Gaussian:

$$\rho(x) = C e^{-ax^2}.$$

Integrals w/ Gaussians:

Let $\displaystyle I = \int_{-\infty}^{+\infty} dx\, e^{-x^2}$

$$\rightarrow I^2 = \int_{-\infty}^{+\infty} dx\, e^{-x^2} \int_{-\infty}^{+\infty} dy\, e^{-y^2} = \int dx\,dy\, e^{-(x^2+y^2)}$$

Switch to polar coordinates $r^2 = x^2 + y^2$

$dx\,dy = 2\pi r\,dr$

$$I^2 = \int_0^\infty 2\pi r\, dr\, e^{-r^2} \quad ; \text{let } u = r^2,\ du = 2r\,dr$$

$$= \pi \int_0^\infty du\, e^{-u} = \pi$$

$$\rightarrow \int_{-\infty}^{+\infty} dx\, e^{-x^2} = \sqrt{\pi}$$

$$\rightarrow \int_{-\infty}^{+\infty} dx\, e^{-ax^2} = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} d(\sqrt{a}\,x)\, e^{-(\sqrt{a}\,x)^2} = \frac{\sqrt{\pi}}{\sqrt{a}}$$

$$\boxed{\int_{-\infty}^{+\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}}$$

For p odd, $\int_{-\infty}^{+\infty} dx\, x^p e^{-ax^2} = 0$

For p even, use $\frac{d}{da} e^{-ax^2} = (-x^2)\, e^{-ax^2}$

$$\rightarrow \int_{-\infty}^{+\infty} dx\, x^{2p}\, e^{-ax^2} = \left(-\frac{d}{da}\right)^p \sqrt{\frac{\pi}{a}}$$

Shifting the Gaussian does not change its area:

$$\int_{-\infty}^{+\infty} dx\, e^{-a(x-b)^2} = \int_{-\infty}^{+\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}\,.$$

<u>How does $\rho(x,t) = |\psi(x,t)|^2$ depend on time?</u>

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}(\psi^* \psi) = \frac{\partial \psi^*}{\partial t}\psi + \psi^* \frac{\partial \psi}{\partial t},$$

but from the Schrodinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi$$

$$-i\hbar \frac{\partial \psi^*}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + V\psi^* \quad \text{(assume } V \text{ is real)}$$

$$\rightarrow \frac{\partial \rho}{\partial t} = \frac{1}{-i\hbar}\left(\frac{-\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + V\psi^*\right)\psi$$

$$+ \psi^* \frac{1}{i\hbar}\left(\frac{-\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi\right)$$

$$= \frac{\hbar}{2mi}\left(\frac{\partial^2 \psi^*}{\partial x^2}\psi - \psi^* \frac{\partial^2 \psi}{\partial x^2}\right)$$

This last line is equal to:

$$= \frac{\hbar}{2mi}\frac{\partial}{\partial x}\left(\frac{\partial \psi^*}{\partial x}\psi - \psi^* \frac{\partial \psi}{\partial x}\right)$$

$$\rightarrow \frac{\partial \rho}{\partial t} = \frac{-\partial j}{\partial x} \quad \text{with} \quad \boxed{j = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x}\psi\right)}$$

or $\boxed{\dfrac{\partial \rho}{\partial t} + \dfrac{\partial j}{\partial x} = 0 \quad \text{Continuity Eq.}}$

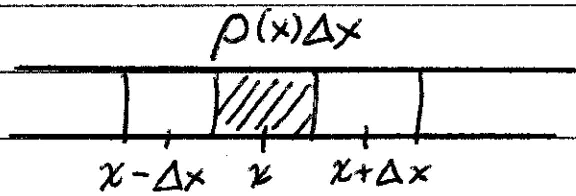
Whenever a quantity is conserved like the total probality,

$$\int_{-\infty}^{+\infty} dx\, \rho(x,t) = 1,$$

there will be a continuity equation.
In 3D the continuity equation is

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0.$$

<u>Physical interpretation:</u> (1D)



In time Δt the current flowing in from the left is $j(x-\frac{\Delta x}{2})\Delta t$, and the current flowing out on the right is $j(x+\frac{\Delta x}{2})\Delta t$.

$$\rightarrow \Delta\text{Probability} = \Delta\rho\,\Delta x = \text{in} - \text{out}$$
$$= j(x-\tfrac{\Delta x}{2})\Delta t - j(x+\tfrac{\Delta x}{2})\Delta t$$
$$\approx -\frac{\partial j}{\partial x}\Delta x\,\Delta t$$

$$\rightarrow \frac{\Delta\rho}{\Delta t} \approx \frac{\partial \rho}{\partial t} \approx -\frac{\partial j}{\partial x}$$