Stationary States

Look for solutions of the Schodinger eq. of the form (separation of variables)

capital Ψ

$$\longrightarrow \Psi(x,t) = \psi(x)\varphi(t)$$

$$i\hbar\frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$$

$$\rightarrow \frac{\partial \Psi}{\partial t} = \psi\frac{d\varphi}{dt} \;;\; \frac{\partial^2\Psi}{\partial x^2} = \frac{d^2\psi}{dx^2}$$

$$\rightarrow i\hbar\psi\frac{d\varphi}{dt} = \left(\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi\right)\varphi$$

$$\rightarrow \underbrace{i\hbar\frac{1}{\varphi}\frac{d\varphi}{dt}}_{\text{only fnt. t}} = \underbrace{\frac{1}{\psi}\left(\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi\right)}_{\text{only fnt. } x} = \underbrace{E}_{\text{const.}}$$

$$i\hbar\frac{1}{\varphi}\frac{d\varphi}{dt} = E \rightarrow \frac{d\varphi}{dt} = -\frac{iE}{\hbar}\varphi$$

$$\rightarrow \varphi(t) = A\exp\left(\frac{-iEt}{\hbar}\right)$$

$$\boxed{\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi}$$

Time Indep. Schrodinger Eq.
ψ = stationary state

The factor A can be included in ψ so that $\boxed{\Psi(x,t) = \psi(x)e^{-iEt/\hbar}}$.

Normalization:

$$|\Psi(x,t)|^2 = |\psi(x)|^2$$

$$\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 dx = \int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1$$

Expectation values:

$$Q(x, \underbrace{\frac{\hbar}{i}\frac{d}{dx}}_{p}) = x,\ p,\ \frac{p^2}{2m}, \ldots \quad \text{any fnt. of } x \ \&\ p$$

$$\langle Q \rangle = \int_{-\infty}^{+\infty} \psi^*(x) e^{\frac{iEt}{\hbar}} Q(x, \frac{\hbar}{i}\frac{d}{dx}) \psi(x) e^{-\frac{iEt}{\hbar}}$$

$$= \int_{-\infty}^{+\infty} \psi^*(x) Q(x, \frac{\hbar}{i}\frac{d}{dx}) \psi(x)$$

= independent of time

Energy:

$$H = \frac{p^2}{2m} + V(x)$$

$$H\psi = E\psi$$

$$H^2\psi = HE\psi = EH\psi = E^2\psi$$

$$\Rightarrow \sigma_H^2 = \langle H^2 \rangle - \langle H \rangle^2 = E^2 - (E)^2 = 0$$

$\Rightarrow$ Measurement yields energy E everytime.

Linear combinations:

how we get time dependent expectation values

If $\Psi_1(x,t) = \psi_1(x)\, e^{-iE_1 t/\hbar}$

& $\Psi_2(x,t) = \psi_2(x)\, e^{-iE_2 t/\hbar}$

are solutions, then so is $c_1\Psi_1 + c_2\Psi_2$.
(linear eq.)

$$\boxed{\psi(x,t) = \sum_n c_n \psi_n(x)\, e^{-iE_n t/\hbar}}$$ is a solution

$$\psi(x,0) = \sum_n c_n \psi_n(x)$$

Completeness:

All solutions can be written in this form if the sum is over all stationary states.

(We will see this explicitly in the next section for a particular problem.)