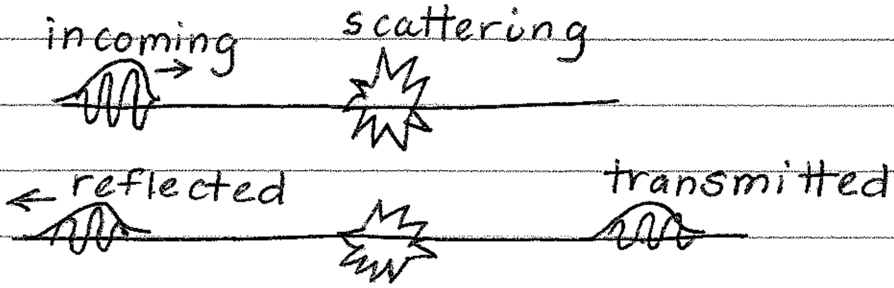
Piecewise Constant Potentials:

Big picture:

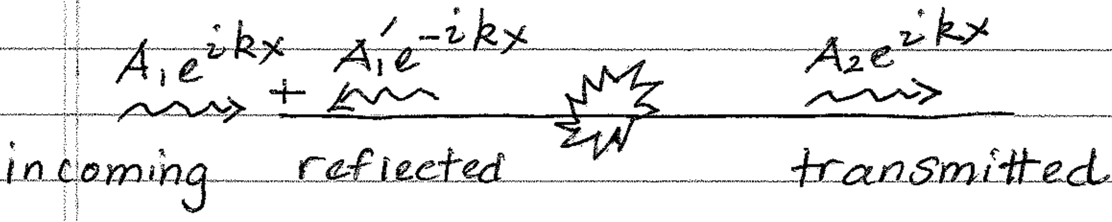
initial:

final:



This is what $\Psi(x,t)$ would look like.
What about the stationary states, $\varphi(x)$?

$$E\varphi = -\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2} + V\varphi$$



We will be solving for $\varphi(x)$ with the understanding that the φ for different E can be combined to form wavepackets.

Constant potential:

$$E\varphi = -\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2} + V\varphi$$

$$\frac{d^2\varphi}{dx^2} = \frac{-2m(E-V)}{\hbar^2}\varphi$$

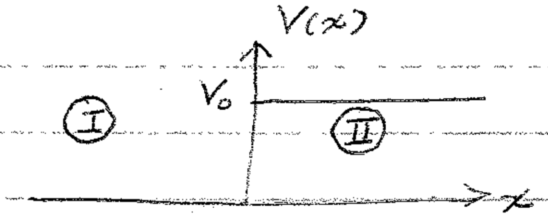
If $E>V$, $\varphi(x) = Ae^{ikx} + A'e^{-ikx}$, $k = \sqrt{\frac{2m(E-V)}{\hbar^2}}$

If $E<V$, $\varphi(x) = Be^{\rho x} + B'e^{-\rho x}$, $\rho = \sqrt{\frac{2m(V-E)}{\hbar^2}}$

Boundary conditions:

$\varphi(x)$ and $\frac{d\varphi}{dx}(x)$ are continuous.

Step Potential:



Suppose $E > V_0$.

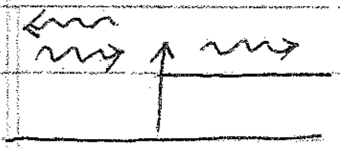
$x < 0, \quad \varphi(x) = A_1 e^{ik_1 x} + A_1' e^{-ik_1 x}, \quad k_1 = \sqrt{\frac{2mE}{\hbar^2}}$

$x > 0, \quad \varphi(x) = A_2 e^{ik_2 x} + A_2' e^{-ik_2 x}, \quad k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

Boundary conditions:

$\varphi(0^-) = \varphi(0^+) \rightarrow A_1 + A_1' = A_2 + A_2'$

$\varphi'(0^-) = \varphi'(0^+) \rightarrow ik_1(A_1 - A_1') = ik_2(A_2 - A_2')$



(wave incoming from L to R)

$A_2' = 0 \rightarrow \quad A_1 + A_1' = A_2$

$\qquad\qquad\quad A_1 - A_1' = \frac{k_2}{k_1} A_2$

$\rightarrow A_1 = \frac{1}{2}\left(1 + \frac{k_2}{k_1}\right) A_2$

$\quad A_1' = \frac{1}{2}\left(1 - \frac{k_2}{k_1}\right) A_2$

$$\rightarrow \frac{A_2}{A_1} = \frac{1}{\frac{1}{2}\left(1 + \frac{k_2}{k_1}\right)} = \frac{2k_1}{k_1 + k_2}$$

$$\frac{A_1'}{A_1} = \frac{A_1'}{A_2}\frac{A_2}{A_1} = \frac{(1 - k_2/k_1)}{1 + k_2/k_1} = \frac{k_1 - k_2}{k_1 + k_2}$$

Probability current:

$|\psi(x)|^2 dx$ = prob finding between x & $x+dx$

$$|\psi(x,t+dt)|^2 dx - |\psi(x,t)|^2 dx =$$

$$= j(x,t)\,dt - j(x+dx,t)\,dt$$

$$\Rightarrow \frac{|\psi(x,t+dt)|^2 - |\psi(x,t)|^2}{dt} = \frac{j(x,t) - j(x+dx,t)}{dx}$$

$$\Rightarrow \frac{\partial|\psi|^2}{\partial t^2} = -\frac{\partial j}{\partial x}$$, j = probability current

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi \qquad < \times\,\psi^*$$

$$-i\hbar\frac{\partial\psi^*}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2} + V\psi^* \qquad < \times\,\psi$$

SUBTRACT:

$$i\hbar\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right) = \frac{-\hbar^2}{2m}\psi^*\frac{\partial^2\psi}{\partial x^2} + \frac{\hbar^2}{2m}\psi\frac{\partial^2\psi^*}{\partial x^2}$$

$$\Rightarrow \frac{\partial|\psi|^2}{\partial t} = \frac{-\hbar}{2mi}\left(\psi^*\frac{\partial^2\psi}{\partial x^2} - \psi\frac{\partial^2\psi^*}{\partial x^2}\right)$$

$$= \frac{-\hbar}{2mi}\frac{\partial}{\partial x}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right)$$

$$\Rightarrow \boxed{j = \frac{\hbar}{2mi}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right)}$$

Example: $\psi(x) = Ae^{ikx} + A'e^{-ikx}$

$$\Rightarrow \qquad \frac{\partial\psi}{\partial x}(x) = ikAe^{ikx} - ikA'e^{-ikx}$$

$$\psi^*(x) = A^*e^{-ikx} + A'^*e^{ikx}$$

$$\frac{\partial\psi^*}{\partial x}(x) = -ikA^*e^{-ikx} + ikA'^*e^{ikx}$$

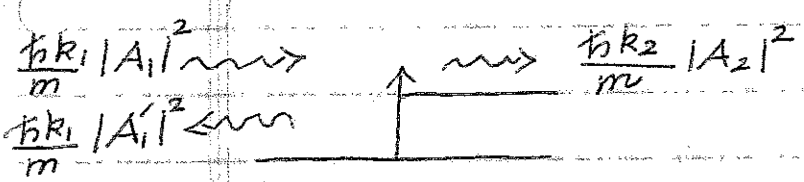
$$\psi^*\frac{\partial\psi}{\partial x} = (A^*e^{-ikx} + A'^*e^{ikx})(ikAe^{ikx} - ikA'e^{-ikx})$$
$$= ik|A|^2 - ik|A'|^2 + ikA'^*Ae^{2ikx} - ikA^*A'e^{-2ikx}$$

$$\psi\frac{\partial\psi^*}{\partial x} = -ik|A|^2 + ik|A'|^2 + ikA'^*Ae^{2ikx} - ikA^*A'e^{-2ikx}$$

$$\Rightarrow \psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x} = 2ik|A|^2 - 2ik|A'|^2$$

$$\Rightarrow j = \frac{\hbar k}{m}|A|^2 - \frac{\hbar k}{m}|A'|^2$$

Apply to step potential:



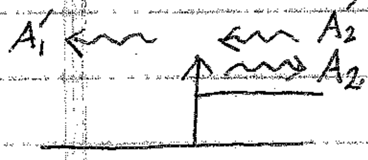
probability currents

$$\frac{j_{refl}}{j_{inc}} = R = \frac{|A_1'|^2}{|A_1|^2} = \left(\frac{k_1-k_2}{k_1+k_2}\right)^2 = \frac{k_1^2-2k_1k_2+k_2^2}{(k_1+k_2)^2} = 1 - \frac{4k_1k_2}{(k_1+k_2)^2}$$

$$\frac{j_{trans}}{j_{inc}} = T = \frac{k_2}{k_1}\frac{|A_2|^2}{|A_1|^2} = \frac{k_2}{k_1}\left(\frac{2k_1}{k_1+k_2}\right)^2 = \frac{4k_1k_2}{(k_1+k_2)^2}$$

$$\Rightarrow \boxed{R+T=1}$$

For completeness I solve here for the case of an incoming wave from the right.



$$A_1 = 0 \rightarrow \quad A_1' = A_2 + A_2'$$

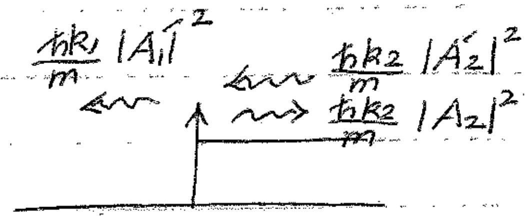
$$-\frac{k_1}{k_2}A_1' = A_2 - A_2'$$

$$\rightarrow \quad A_2 = \tfrac{1}{2}\left(1-\frac{k_1}{k_2}\right)A_1'$$

$$A_2' = \tfrac{1}{2}\left(1+\frac{k_1}{k_2}\right)A_1'$$

$$\frac{A_2}{A_2'} = \frac{1-k_1/k_2}{1+k_1/k_2} = \frac{k_2-k_1}{k_2+k_1}$$

$$\frac{A_1'}{A_2'} = \frac{2}{1+k_1/k_2} = \frac{2k_2}{k_1+k_2}$$



$$R = \frac{|A_2|^2}{|A_2'|^2} = \frac{(k_2-k_1)^2}{(k_2+k_1)^2} = 1 - \frac{4k_1k_2}{(k_1+k_2)^2}$$

$$T = \frac{|A_1'|^2}{|A_2'|^2}\frac{k_1}{k_2} = \frac{k_1}{k_2}\left(\frac{2k_2}{k_1+k_2}\right)^2 = \frac{4k_1k_2}{(k_1+k_2)^2} \qquad \checkmark\ R+T=1$$