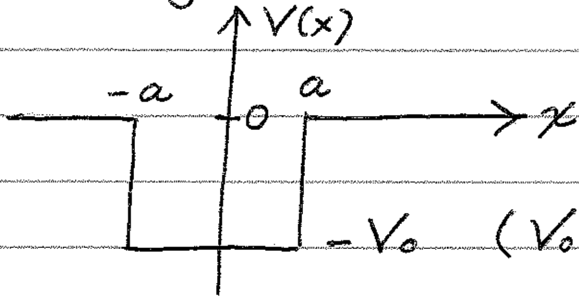
Potential Well:

Following the conventions in Griffiths



$-V_0 < E < 0$

$(V_0 > 0$ so $-V_0 < 0)$

This potential is symmetric: $V(x) = V(-x)$.
Let ψ be a solution to the time independent Schrodinger equation:

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx} + V(x)\psi(x) = E\,\psi(x).$$

Then let $f(x) = \psi(-x)$

$$f'(x) = -\psi'(-x)$$

$$f''(x) = \psi''(-x).$$

$$-\frac{\hbar^2}{2m}\psi''(-x) + V(-x)\psi(-x) = E\,\psi(x)$$

$$\rightarrow \frac{-\hbar^2}{2m} f''(x) + V(-x) f(x) = E f(x)$$

$$\rightarrow -\frac{\hbar^2}{2m} f''(x) + V(x) f(x) = E f(x)$$

$\Rightarrow f(x) = \psi(-x)$ is also a solution with the same energy.

$\rightarrow$ $\psi(x)+\psi(-x)$ is a solution, $\leftarrow$ even
and $\psi(x)-\psi(-x)$ is a solution. $\leftarrow$ odd

$\rightarrow$ We can assume the solution is either even or odd. It can not be both because then $\psi=0$.

Even case:

$$\varphi(x>a) = F e^{-\rho x} \quad , \quad \rho = \sqrt{\frac{2m(-E)}{\hbar^2}}$$

$$\varphi(x<-a) = F e^{\rho x}$$

$$\varphi(-a<x<a) = D\cos(kx) \qquad k = \sqrt{\frac{2m(E--V_0)}{\hbar^2}}$$

Boundary Conditions:
$$\varphi(a) = F e^{-\rho a} = D\cos(ka)$$
$$\varphi'(a) = -\rho F e^{-\rho a} = -k D\sin(ka)$$

$$\rightarrow \quad -\rho = -k\frac{\sin(ka)}{\cos(ka)} \rightarrow \boxed{\tan(ka) = \frac{\rho}{k}}$$

Odd case:

$$\varphi(x>a) = F e^{-\rho x}$$

$$\varphi(x<-a) = -F e^{\rho x}$$

$$\varphi(-a<x<a) = D \sin(kx)$$

Boundary Conditions: $\varphi(a) = F e^{-\rho a} = D \sin(ka)$

$\varphi'(a) = -\rho F e^{-\rho a} = kD\cos(ka)$

$$\rightarrow -\rho = k\frac{\cos(ka)}{\sin(ka)} \longrightarrow \boxed{\cot(ka) = -\frac{\rho}{k}}$$

Graphical solution:

Note that $\rho^2 + k^2 = \dfrac{2mV_0}{\hbar^2}$

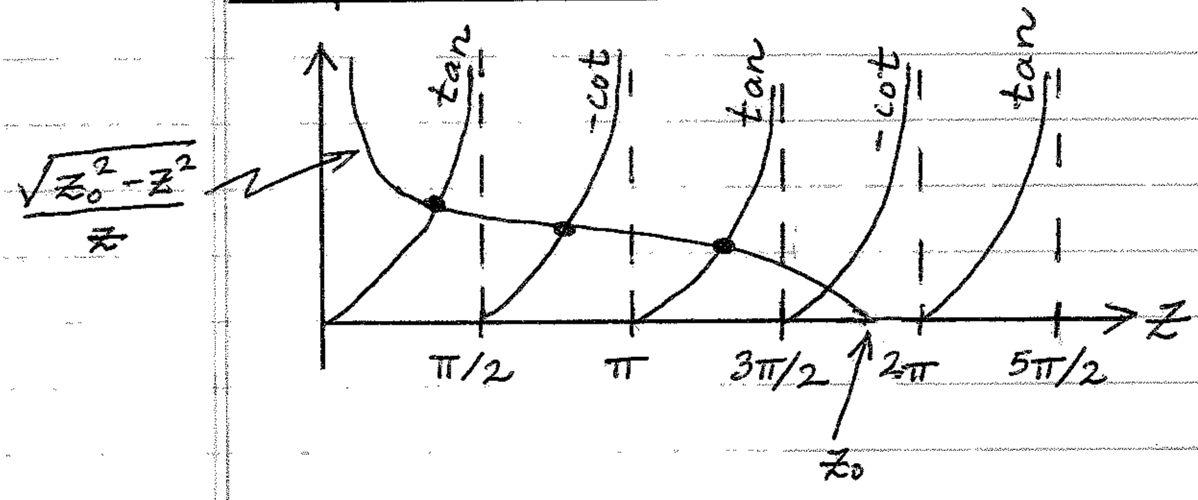
$$\rightarrow (\rho a)^2 + \underbrace{(ka)^2}_{\equiv z^2} = \underbrace{\frac{2mV_0 a^2}{\hbar^2}}_{z_0^2}$$

Defining $z = ka$ and $z_0 = \sqrt{\dfrac{2mV_0 a^2}{\hbar^2}}$ as in Griffiths:

$$\boxed{\begin{array}{ll} \text{Even:} & \tan(z) = \dfrac{\sqrt{z_0^2 - z^2}}{z} \\ \text{Odd:} & -\cot(z) = \dfrac{\sqrt{z_0^2 - z^2}}{z} \end{array}}$$

note that $z^2 \propto E + V_0$.

Graphical solution:



★ The intersections are the solutions (•).

★ There is always at least one solution.

★ For $n\frac{\pi}{2} < z_0 < (n+1)\frac{\pi}{2}$ there are $n+1$ solutions.

★ The solutions alternate: even, odd, even, odd, ...

★ If V_0 is very large, the solutions at least for small z occur near

$$ka = z \approx n\frac{\pi}{2} \quad \text{for } n = 1, 2, 3, \ldots$$

$$\rightarrow k = \frac{n\pi}{2a} \quad \text{and} \quad E + V_0 = \frac{\hbar^2}{2m}\left(\frac{n\pi}{2a}\right)^2.$$

★ Remembering that $2a$ is the well width, this is the infinite square well result.