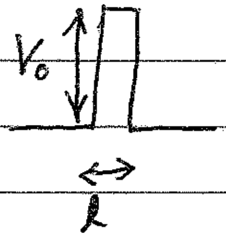
Delta Function Potential:



What happens if we take $V_0 \to \infty$, $\ell \to 0$ in such a way that $V_0 \ell = \alpha$?

$$\boxed{V(x) = \alpha\, \delta(x)}$$

From the potential barrier we know that

$$T = \frac{1}{1 + \frac{(k^2+\rho^2)^2}{4\rho^2 k^2} \sinh^2(\rho \ell)},$$

where $E = \frac{\hbar^2 k^2}{2m}$ and $\rho = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$.

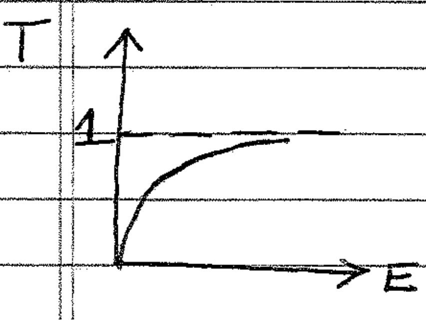
As $V_0 \to \infty$, $\rho^2 \ell \to \frac{2m V_0 \ell}{\hbar^2} = \frac{2m\alpha}{\hbar^2}$.

Because $\rho \to \infty$, $\frac{\rho^2 \ell}{\rho} = \rho \ell \to 0$.

$\sinh(\rho\ell) \to \rho\ell$ and

$$T \to \frac{1}{1 + \frac{\rho^4}{4\rho^2 k^2}(\rho\ell)^2} = \frac{1}{1 + \frac{1}{4k^2}(\rho^2 \ell)^2}$$

$$= \boxed{\frac{1}{1 + \frac{1}{4}\frac{2m\alpha^2}{\hbar^2 E}} = T}$$



Working w/ delta fnt. potentials:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \alpha\delta(x)\psi = E\psi$$

Integrate from $-\varepsilon$ to ε:

$$\int_{-\varepsilon}^{+\varepsilon} \frac{d^2\psi}{dx^2}\,dx = \frac{d\psi}{dx}(\varepsilon) - \frac{d\psi}{dx}(-\varepsilon)$$

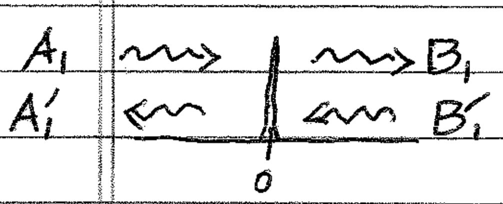
$$\int_{-\varepsilon}^{+\varepsilon} \delta(x)\psi\,dx = \psi(0)$$

$$\int_{-\varepsilon}^{+\varepsilon} E\psi\,dx \approx E\psi(0)\cdot 2\varepsilon \to 0 \text{ as } \varepsilon \to 0$$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\left(\frac{d\psi}{dx}(0^+) - \frac{d\psi}{dx}(0^-)\right) + \alpha\psi(0) = 0$$

$$\Rightarrow \quad \boxed{\frac{d\psi}{dx}(0^+) - \frac{d\psi}{dx}(0^-) = \frac{2m\alpha}{\hbar^2}\psi(0)}$$

The derivative is discontinuous.



$$\varphi(x<0) = A_1 e^{ikx} + A_1' e^{-ikx}$$
$$\varphi(x>0) = B_1 e^{ikx} + B_1' e^{-ikx}, \quad k = \sqrt{\frac{2mE}{\hbar^2}}$$

Boundary conditions:

$$\varphi(0) = A_1 + A_1' = B_1 + B_1'$$
$$\frac{d\varphi}{dx}(0^+) - \frac{d\varphi}{dx}(0^-) = ik(B_1 - B_1' - A_1 + A_1') = \frac{2m\alpha}{\hbar^2}\varphi(0)$$

Consider the case of incoming from the left: $B_1' = 0$.

$$A_1 + A_1' = B_1$$
$$A_1 - A_1' - B_1 = \frac{-2m\alpha}{ik\hbar^2} B_1$$

$$A_1 - A_1' = B_1 - \frac{2m\alpha}{ik\hbar^2} B_1$$

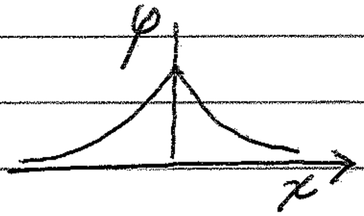
$\rightarrow$ $2A_1 = \left(2 - \frac{2m\alpha}{ik\hbar^2}\right)B_1$ and $2A_1' = +\frac{2m\alpha}{ik\hbar^2} B_1$

$\rightarrow$ $T = \frac{|B_1|^2}{|A_1|^2} = \left|\frac{1}{1 - \frac{m\alpha}{ik\hbar^2}}\right|^2, \quad R = \frac{|A_1'|^2}{|A_1|^2} = \left|\frac{+\frac{m\alpha}{ik\hbar^2}}{1 - \frac{m\alpha}{ik\hbar^2}}\right|^2$

$$\rightarrow T = \frac{1}{1 + \left(\frac{m\alpha}{\hbar^2 k}\right)^2} \quad \text{and} \quad R = \frac{\left(\frac{m\alpha}{\hbar^2 k}\right)^2}{1 + \left(\frac{m\alpha}{\hbar^2 k}\right)^2}$$

But $\left(\frac{m\alpha}{\hbar^2 k^2}\right)^2 = \frac{m\alpha^2}{\hbar^2}\frac{m}{\hbar^2 k^2}\frac{2}{2} = \frac{m\alpha^2}{2\hbar^2 E}$

$$\rightarrow \boxed{T = \frac{1}{1 + \frac{m\alpha^2}{2\hbar^2 E}}}$$ as on page 1.



What happens if $E < 0$?

$$\left.\begin{array}{l} \varphi(x<0) = Ae^{+\rho x} \\ \varphi(x>0) = Be^{-\rho x} \end{array}\right\} \text{to be normalizable}, \quad \rho = \sqrt{\frac{2m(-E)}{\hbar^2}}$$

Boundary conditions:

$\varphi(0) = A = B$

$$\frac{d\varphi}{dx}(0^+) - \frac{d\varphi}{dx}(0^-) = \rho(-B - A) = -\rho\underbrace{(A+B)}_{2A} = \frac{2m\alpha}{\hbar^2}\underbrace{\varphi(0)}_{A}$$

$$\rightarrow 2\rho = -\frac{2m\alpha}{\hbar^2} \rightarrow \rho = -\frac{m\alpha}{\hbar^2} \rightarrow \rho^2 = \frac{-2mE}{\hbar^2} = \left(\frac{m}{\hbar^2}\right)^2\alpha^2$$

$$\rightarrow \boxed{E = -\frac{1}{2}\frac{m\alpha^2}{\hbar^2}}$$ This is the only energy for which there is a solution for $E < 0$. Note that $\alpha = -\frac{\hbar^2}{m}\rho < 0$. Also, normalization implies $A = B = \sqrt{\rho}$.