Name:

Exam 1 - PHY 4604 - Fall 2019
October 2, 2019
8:20-10:10PM, CSE E119

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

Unless otherwise noted all parts (a), (b), ... are worth 5 points, and the entire exam is 100 points.

Harmonic oscillator:

$$
\begin{aligned}
a_+ &= \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x) \\
a_- &= \frac{1}{\sqrt{2\hbar m\omega}}(+ip + m\omega x) \\
x &= \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \\
p &= i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-) \\
\psi_0(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\
\psi_n &= \frac{1}{\sqrt{n!}}(a_+)^n \psi_0
\end{aligned}
$$

Delta function potential $V(x) = \alpha\,\delta(x)$:

$$\frac{d\varphi(0^+)}{dx} - \frac{d\varphi(0^-)}{dx} = \frac{2m\alpha}{\hbar^2}\varphi(0).$$

1. **Short answer section**

(a) Write down the time dependent Schrodinger equation in one dimension.

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi$$

(b) What is the uncertainty in x, $\sigma_x = \Delta x$, in terms of the expectation values of x and x^2?

$$\sigma_x = \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

(c) For a one dimensional system with $V = 0$, the wave function as a function of time is

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \phi(k)\, e^{ikx} e^{-i\frac{\hbar k^2}{2m}t}. \quad (1)$$

What is $\phi(k)$ in terms of $\psi(x,0)$?

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, \psi(x,0)\, e^{-ikx}$$

(d) What is the solution to the time independent Schrodinger equation with a constant potential $V(x) = V_o$ and $E > V_o$? Define the relevant constants in terms of E, V_o, $\hbar$, ...

$$\psi(x) = A e^{ikx} + A' e^{-ikx}, \quad k = \sqrt{\frac{2m(E - V_o)}{\hbar^2}}$$

(e) What is the uncertainty principle?

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

2. **General properties**
Consider the wave function $\psi(x) = -C$ for $0 \leq x < 1/2$ and $\psi(x) = C$ for $1/2 < x \leq 1$. The constant C is a positive real number, and the function is zero everywhere other than $0 \leq x \leq 1$.

(a) What is the constant C so that the wave function is normalized?

$$1 = \int_0^1 dx\, |\psi(x)|^2 = \int_0^1 dx\, C^2 = C^2 \rightarrow \boxed{C = 1}$$

(b) Compute the expectation value of $\langle x \rangle$.

$$\langle x \rangle = \int_0^1 dx\, x\, \underbrace{|\psi(x)|^2}_{1} = \frac{1}{2}$$

(c) Compute the expectation value of x^2 and determine Δx. Based on the uncertainty principle, what is a lower bound on Δp?

$$\langle x^2 \rangle = \int_0^1 dx\, x^2\, \underbrace{|\psi(x)|^2}_{1} = \frac{1}{3}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{1}{3} - \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}}$$

$$\Delta x \Delta p \geq \frac{\hbar}{2} \longrightarrow \Delta p \geq \sqrt{3}\,\hbar$$

(d) Determine the coefficients, c_n, when $\psi(x)$ is expressed in terms of the eigenstates, $\psi_n(x)$ of the infinite square well for $0 \le x \le 1$:

$\psi(x) = -1$

$$\psi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x). \quad (2)$$

$\psi(x) = 1$

$$c_n = -\int_0^{1/2} dx \sqrt{2} \sin(n\pi x) + \int_{1/2}^{1} dx \sqrt{2} \sin(n\pi x)$$

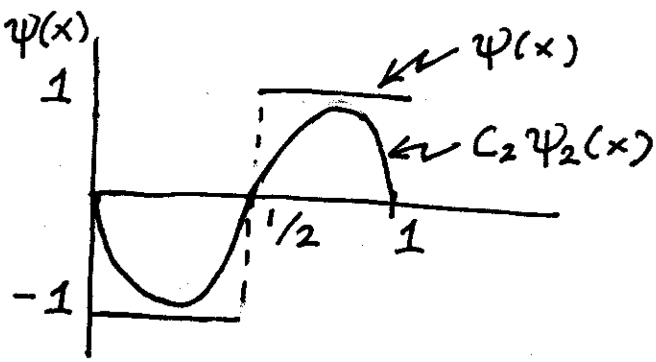
$$= \frac{\sqrt{2}\cos(n\pi x)}{n\pi}\Big|_0^{1/2} - \frac{\sqrt{2}\cos(n\pi x)}{n\pi}\Big|_{1/2}^{1}$$

$$\boxed{c_n = \frac{\sqrt{2}}{n\pi}\left(2\cos\left(\frac{n\pi}{2}\right) - 1 - \cos(n\pi)\right)}$$

(e) What is the value of the first non-zero c_n? For this term sketch $c_n\phi_n(x)$ and the original $\psi(x)$ on the same graph below. Make sure to label your axes.

$c_1 = 0$

$$c_2 = \frac{\sqrt{2}}{2\pi}(-2-1-1) = -\frac{2\sqrt{2}}{\pi} \approx -\frac{2.8}{3.1} \approx -0.9$$



3. **Harmonic oscillator**

At $t = 0$ the wave function of a particle in a harmonic oscillator potential is given by

$$\psi(x,0) = \frac{i}{\sqrt{3}}(\psi_0(x) + \psi_1(x) + \psi_2(x)).$$

(a) What is $\psi(x,t)$?

$$\psi(x,t) = \frac{i}{\sqrt{3}}\left(\psi_0(x)e^{-i\frac{\omega}{2}t} + \psi_1(x)e^{-i\frac{3\omega}{2}t} + \psi_2(x)e^{-i\frac{5}{2}\omega t}\right)$$

(b) What is the expectation value of $\langle p\rangle$ as a function of time?

$$\langle p\rangle = \frac{1}{3}\int dx\left(\psi_0^*(x)e^{i\frac{\omega}{2}t} + \psi_1^*(x)e^{i\frac{3\omega}{2}t} + \psi_2^*(x)e^{i\frac{5\omega}{2}t}\right)$$

$$i\sqrt{\frac{\hbar m\omega}{2}}\,(a_+ - a_-)$$

$$\left(\psi_0(x)e^{-i\frac{\omega}{2}t} + \psi_1(x)e^{-i\frac{3\omega}{2}t} + \psi_2(x)e^{-i\frac{5\omega}{2}t}\right)$$

$$= \frac{i}{3}\sqrt{\frac{\hbar m\omega}{2}}\left(\underbrace{\sqrt{1}\,e^{i\omega t}}_{\text{from } a_+} + \underbrace{\sqrt{2}\,e^{i\omega t}}_{\text{from } a_+} - \underbrace{\sqrt{1}\,e^{-i\omega t}}_{\text{from } a_-} - \underbrace{\sqrt{2}\,e^{-i\omega t}}_{\text{from } a_-}\right)$$

$$\boxed{\langle p\rangle = -\frac{2}{3}\sqrt{\frac{\hbar m\omega}{2}}\,(1+\sqrt{2})\sin\omega t}$$

$$\langle p^2\rangle = \frac{1}{3}\int dx\left(\psi_0^*(x)e^{i\frac{\omega}{2}t} + \psi_1^*(x)e^{i\frac{3\omega}{2}t} + \psi_2^*(x)e^{i\frac{5\omega}{2}t}\right)$$

$$\frac{\hbar m\omega}{2}\left(a_+a_- + a_-a_+ - (a_+)^2 - (a_-)^2\right)$$

$$\left(\psi_0(x)e^{-i\frac{\omega}{2}t} + \psi_1(x)e^{-i\frac{3\omega}{2}t} + \psi_2(x)e^{-i\frac{5\omega}{2}t}\right)$$

Using $(a_+a_- + a_-a_+)\psi_n = (2n+1)\psi_n$,

(c) What are the expectation values of $\langle p^2 \rangle$ as a function of time?

(from previous page)

$$\langle p^2 \rangle = \frac{1}{3}\frac{\hbar m\omega}{2}\Big(\underbrace{2\cdot 0+1 + 2\cdot 1+1 + 2\cdot 2+1}_{\text{from } a_+a_- + a_-a_+} \underbrace{-\sqrt{1}\sqrt{2}}_{(a_+)^2} e^{-i2\omega t} \underbrace{-\sqrt{1}\sqrt{2}}_{(a_-)^2} e^{i2\omega t}\Big)$$

$$= \frac{1}{3}\frac{\hbar m\omega}{2}\left(9 - 2\sqrt{2}\cos(2\omega t)\right)$$

(d) What is the expectation value of the energy as a function of time?

$$\langle E \rangle = \sum_n |c_n|^2 E_n = \frac{1}{3}\left(0+\frac{1}{2} + 1+\frac{1}{2} + 2+\frac{1}{2}\right)\hbar\omega = \frac{3}{2}\hbar\omega$$

(e) Evaluate the commutator $[a_-, a_+a_-a_+]$.

$$[a_-, a_+a_-a_+] = [a_-, a_+]a_-a_+ + 0 + a_+a_-[a_-, a_+]$$

$$= a_-a_+ + a_+a_-$$

4. **Piecewise constant potentials**

For this problem consider the one dimensional time independent Schrodinger equation with $V(x) = \alpha\delta(x) + V_o\theta(x)$, where α and V_o are positive real numbers. $\theta(x)$ is 1 for $x > 0$ and 0 otherwise. **Take the energy of the particle to be less than V_o: $E < V_o$.**

(a) The solution to the time independent Schrodinger equation in the regions $x < 0$ has the form

$$\psi(x < 0) = A_1 e^{ik_1 x} + A_1' e^{-ik_1 x}. \tag{3}$$

What is k_1 in terms of E, V_o, $\hbar$, and m?

$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$

(b) What is the form of the **physical** solution for $x > 0$. Define the constants determined by E, V_o, $\hbar$, and m.

$B e^{-\rho x}$, $\rho = \sqrt{\frac{2m(V_o - E)}{\hbar^2}}$

(c) Write down the boundary conditions at $x = 0$?

$$\boxed{\begin{aligned} &A_1 + A_1' = B \\ &\rho B - i k_1 (A_1 - A_1') = \frac{2m\alpha}{\hbar^2} B \end{aligned}}$$

$$A_1 - A_1' = \frac{1}{-i k_1}\left(-\rho + \frac{2m\alpha}{\hbar^2}\right) B = i\left(\frac{\rho}{k_1} - \frac{2m\alpha}{\hbar^2}\right) B$$

$$\rightarrow 2A_1 = \left\{1 + i\left(\frac{\rho}{k_1} - \frac{2m\alpha}{\hbar^2}\right)\right\} B$$

$$2A_1' = \left\{1 - i\left(\frac{\rho}{k_1} - \frac{2m\alpha}{\hbar^2}\right)\right\} B$$

(d) Solve for the ratio A_1'/A_1.

$$\frac{A_1'}{A_1} = \frac{1 - i\left\{\frac{\rho}{k_1} - \frac{2m\alpha}{\hbar^2}\right\}}{1 + i\left\{\frac{\rho}{k_1} - \frac{2m\alpha}{\hbar^2}\right\}}$$

(e) Using the result of part (d), what is the reflection probability?

$$R = \left|\frac{A_1'}{A_1}\right|^2 = 1$$

note magnitude squared