Solution

Name:

Exam 2 - PHY 4604 - Fall 2019
November 13, 2019
8:20-10:10PM, CSE E119

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

Unless otherwise noted, all parts (a), (b), … are worth 5 points, and the entire exam is 100 points.

$$
\begin{aligned}
Y_0^0 &= \frac{1}{2}\sqrt{\frac{1}{\pi}} \\
Y_1^1(\theta,\varphi) &= \frac{-1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta \\
Y_1^0(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{\pi}}\cdot\cos\theta \\
Y_1^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta \\
Y_2^2(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot e^{2i\varphi}\cdot\sin^2\theta \\
Y_2^1(\theta,\varphi) &= \frac{-1}{2}\sqrt{\frac{15}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta\cdot\cos\theta \\
Y_2^0(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{5}{\pi}}\cdot(3\cos^2\theta-1) \\
Y_2^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{15}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta\cdot\cos\theta \\
Y_2^{-2}(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot e^{-2i\varphi}\cdot\sin^2\theta
\end{aligned}
$$

Delta function potential $V(x)=\alpha\,\delta(x)$:

$$\frac{d\varphi(0^+)}{dx}-\frac{d\varphi(0^-)}{dx}=\frac{2m\alpha}{\hbar^2}\,\varphi(0).$$

1. **Short Answer Section**

(a) What is the definition of the adjoint, $A^\dagger$, of an operator A?

$$\langle A^\dagger f | g \rangle = \langle f | A g \rangle$$

(b) If two operators commute, what can be said about their eigenvalues and/or eigenvectors?

One can find a set of eigenvectors that work for both operators.

(c) If the state of the system at time t is given by $|\psi(t)\rangle$, what is the probability of measuring a non-degenerate eigenvalue a of operator A with eigenvector $|\psi_a\rangle$?

$$|\langle \psi_n | \psi(t) \rangle|^2$$

(d) What is the degeneracy of the eigenstates of the Hydrogen atom eigenstates with energy $-13.6eV/3^2$? Include the spin degeneracy in your calculation.

		degeneracy
For $n=3$, $\ell=2$	$m=-2,\ldots,2$	2×5
$\ell=1$	$m=-1,0,1$	2×3
$\ell=0$	$m=0$	2×1
		$\boxed{18}$

2. **General properties**

Consider the operator

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2i \\ 0 & -2i & 0 \end{pmatrix}. \quad (1)$$

(a) What are the eigenvalues of A?

$$0 = \det(A - \lambda \mathbb{1}) = \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 2i \\ 0 & -2i & -\lambda \end{pmatrix}$$

$$= \lambda^2 (1-\lambda) - (2i)(-2i)(1-\lambda)$$

$$= \lambda^2 (1-\lambda) - 4(1-\lambda)$$

$$= (\lambda^2 - 4)(1-\lambda)$$

$$\rightarrow \lambda = 1, +2, -2$$

(b) What are the normalized eigenvectors for these eigenvalues?

$\lambda = 1$: eigenvector $= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = 2$: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2i \\ 0 & -2i & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 2 \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \rightarrow$ $c_1 = 2c_1 \rightarrow c_1 = 0$, $\cancel{2} i c_3 = \cancel{2} c_2$

$\rightarrow$ eigenvector $= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$

$\lambda = -2$: eigenvector $= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$ by orthogonality

(c) Let the eigenvectors be denoted by $|\psi_a\rangle$, $|\psi_b\rangle$ and $|\psi_c\rangle$. What are the projection operators $|\psi_a\rangle\langle\psi_a|$, $|\psi_b\rangle\langle\psi_b|$ and $|\psi_c\rangle\langle\psi_c|$?

$$\lambda = 1: \quad |\psi_a\rangle\langle\psi_a| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda = 2: \quad |\psi_b\rangle\langle\psi_b| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix}$$

$$\lambda = -2: \quad |\psi_c\rangle\langle\psi_c| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & 1 \end{pmatrix}$$

(d) Find an operator, B, which has the same eigenvectors as A, but different eigenvalues. B can not be proportional to either the identity matrix or A. It can also not have an zero eigenvalues.

$$B = \alpha|\psi_a\rangle\langle\psi_a| + \beta|\psi_b\rangle\langle\psi_b| + \gamma|\psi_c\rangle\langle\psi_c|$$

since A & B have the same eigenvectors.

For example, if $\alpha = 3$, $\beta = 2$, $\gamma = 1$, then

$$B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3/2 & i/2 \\ 0 & -i/2 & 3/2 \end{pmatrix}.$$

3. **Measurements**

Three Hermitian operators have eigenvalues, λ, and eigenvectors are given below.

$$\sigma_x \text{ operator} \;:\; \lambda = 1 \text{ with } \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \text{ and } \lambda = -1 \text{ with } \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$
$$\sigma_y \text{ operator} \;:\; \lambda = 1 \text{ with } \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \text{ and } \lambda = -1 \text{ with } \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$
$$\sigma_z \text{ operator} \;:\; \lambda = 1 \text{ with } \begin{pmatrix}1\\0\end{pmatrix} \text{ and } \lambda = -1 \text{ with } \begin{pmatrix}0\\1\end{pmatrix}$$

(a) At $t = 0$ a measurement of σ_y is made, and the result is -1. What is the state of the system immediately after this measurement.

$$|\psi(0^+)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

(b) If the Hamiltonian of the system is $H = \hbar\omega\sigma_x$, what is the state of the system at time t after the measurement in part (a)?

$$|\psi(0^+)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}\right\} = \frac{1-i}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$
$$+ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}\right\} + \frac{1+i}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

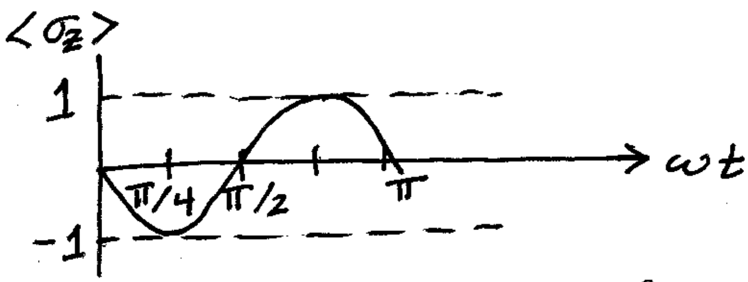
$$\rightarrow |\psi(t)\rangle = e^{-i\omega t}\frac{1-i}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} + e^{i\omega t}\frac{1+i}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}\cos\omega t - \sin\omega t\\ -i\cos\omega t - i\sin\omega t\end{pmatrix}$$

(c) At time t a measurement of σ_z is performed. What are the possible outcomes of this measurement and the associated probabilities?

Possible outcomes	Probability
$+1$	$\lvert\langle\lambda=1\vert\psi(t)\rangle\rvert^2 = \frac{1}{2}\lvert\cos\omega t - \sin\omega t\rvert^2 = \frac{1}{2}(1-2\sin\omega t\cos\omega t) = \frac{1}{2}(1-\sin 2\omega t)$ for σ_z
-1	$\lvert\langle\lambda=-1\vert\psi(t)\rangle\rvert^2 = \frac{1}{2}\lvert -i\cos\omega t - i\sin\omega t\rvert^2 = \frac{1}{2}(\cos\omega t + \sin\omega t)^2 = \frac{1}{2}(1+2\sin\omega t\cos\omega t) = \frac{1}{2}(1+\sin 2\omega t)$ for σ_z

(d) If many measurements of σ_z as in part (c) are made, what is the average value of σ_z? Plot this expectation values of σ_z as a function of ωt.

$$\langle\sigma_z\rangle = 1\cdot \text{Prob.}(\lambda=1) + (-1)\cdot\text{Prob.}(\lambda=-1)$$
$$= \frac{1}{2}(1-\sin 2\omega t) - \frac{1}{2}(1+2\sin\omega t)$$
$$= -\sin 2\omega t$$



4. **Angular momentum**

(a) What is the commutator $[L_x L_y L_z, L_z]$?

$$[L_x L_y L_z, L_z] = L_x L_y L_z L_z - L_z L_x L_y L_z$$

$$= [L_x L_y, L_z] L_z$$

$$= \underbrace{[L_x, L_z]}_{-i\hbar L_y} L_y L_z + L_x \underbrace{[L_y, L_z]}_{i\hbar L_x} L_z$$

$$= i\hbar (L_x^2 - L_y^2) L_z$$

(b) What is the matrix element $\langle 2,1|L_z L_x|2,0\rangle$?

$$\langle 2,1|L_z L_x|2,0\rangle = \langle 2,1| L_z \frac{L_+ + L_-}{2} |2,0\rangle$$

$$= \langle 2,1| L_z \frac{L_+}{2} |2,0\rangle$$

$$= \frac{1}{2} \langle 2,1| L_z \hbar \sqrt{2(2+1) - 0(0+1)}\, |2,1\rangle$$

$$= \frac{\sqrt{6}}{2} \hbar \langle 2,1| L_z |2,1\rangle$$

$$= \frac{\sqrt{6}}{2} \hbar^2$$

(c) Evaluate the non-zero matrix elements of $\langle \frac{3}{2}, m|(J_z)^2|\frac{3}{2}, m'\rangle$.

$$\langle \tfrac{3}{2}, \tfrac{3}{2} | J_z^2 | \tfrac{3}{2}, \tfrac{3}{2} \rangle = \tfrac{9}{4}\hbar^2$$

$$\langle \tfrac{3}{2}, \tfrac{1}{2} | J_z^2 | \tfrac{3}{2}, \tfrac{1}{2} \rangle = \tfrac{1}{4}\hbar^2$$

$$\langle \tfrac{3}{2}, -\tfrac{1}{2} | J_z^2 | \tfrac{3}{2}, \tfrac{-1}{2} \rangle = \tfrac{1}{4}\hbar^2$$

$$\langle \tfrac{3}{2}, \tfrac{-3}{2} | J_z^2 | \tfrac{3}{2}, \tfrac{-3}{2} \rangle = \tfrac{9}{4}\hbar^2$$

(d) In spherical coordinates a wave function has the form $\psi(r, \theta, \phi) = R(r)f(\theta, \phi)$. Find a function $f(\theta, \phi)$ for which an L^2 measurement yields $2\hbar^2$ 50% of the time and $6\hbar^2$ 50% of the time, but an L_z measurement always yields $\hbar$.

$$\frac{1}{\sqrt{2}} Y_{1,1} + \frac{1}{\sqrt{2}} Y_{2,1}$$

$\ell = 1 \quad m = 1 \qquad \ell = 2 \quad m = 1$

5. **Radial Schrodinger equation**

A particle moves in the radial potential given by $V(r) = -V_o\delta(r-a)$, where $a > 0$ and $V_o > 0$. In the following take $E < 0$ and $l = 0$.

(a) What is the solution to the $u(r)$ radial equation for $0 < r < a$ with $E < 0$ and $l = 0$? Take into account the behavior of $u(r)$ as $r \to 0$ to eliminate any unphysical solutions.

$$u(0) = 0$$

$$u(r) = A\sinh(\rho r) \ , \ \rho = \sqrt{\frac{2m(-E)}{\hbar^2}}$$

(b) What is the solution to the $u(r)$ radial equation for $r > a$ with $E < 0$ and $l = 0$? Take into account the behavior of $u(r)$ as $r \to \infty$ to eliminate any unphysical solutions.

$u(r) = Be^{-\rho r}$ since $u \to 0$ as $r \to \infty$.

(c) Write down the equations for the boundary conditions at $r = a$. (Hint: Look at page 1 of the exam.)

$$\star \quad u(a) = A\sinh(\rho a) = Be^{-\rho a}$$

$$u'(a^+) - u'(a^-) = \frac{2m(-V_0)}{\hbar^2}\, u(a)$$

$$\rightarrow -\rho B e^{-\rho a} - A\rho\cosh(\rho a) = \frac{-2mV_0}{\hbar^2} B e^{-\rho a}$$

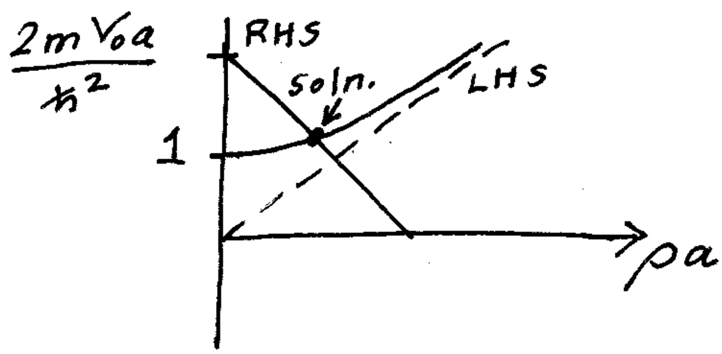
$$\star \quad \rightarrow \quad -A\rho\cosh(\rho a) = \left(\rho - \frac{2mV_0}{\hbar^2}\right) B e^{-\beta}$$

(d) Solve the boundary conditions in part (c) graphically. What is the condition that there be at least one bound state solution?

Divide the ★ eqs. above:

$$-\rho\,\frac{\cosh(\rho a)}{\sinh(\rho a)} = \rho - \frac{2mV_0}{\hbar^2}$$

$$\rightarrow \quad \rho a\,\frac{\cosh(\rho a)}{\sinh(\rho a)} = \frac{2mV_0 a}{\hbar^2} - \rho a$$



There is one bound state as long as $\frac{2mV_0 a}{\hbar^2} > 1$.