Name: Solution

Final Exam - PHY 4604 - Fall 2019
December 9, 2019
3:00P-5:00P, NPB 1002

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

Each exam question, (a),..., (d), is worth 5 points, and the entire exam is out of 100 points. Some formulas are given with the relevant question.

For the harmonic oscillator

$$\begin{aligned} x &= \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \\ p &= i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-). \end{aligned}$$

Delta function potential $V(x) = \alpha\,\delta(x)$:

$$\frac{d\varphi(0^+)}{dx} - \frac{d\varphi(0^-)}{dx} = \frac{2m\alpha}{\hbar^2}\,\varphi(0).$$

1. **Short answer:**

(a) Write down the time dependent Schrodinger equation.

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \nabla^2 \psi + V\psi \quad \ldots 3D$$

or

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V\psi \quad \ldots 1D$$

(b) Write down the wave function, $\psi(r_1, r_2)$, for Fermions in single particles states $\phi_a(r)$ and $\phi_b(r)$.

$$\psi(r_1, r_2) = \frac{1}{\sqrt{2}} \left(\varphi_a(r_1)\varphi_b(r_2) - \varphi_a(r_2)\varphi_b(r_1) \right)$$

(c) Write down an expression for the probability current in one dimension.

$$j = \frac{\hbar}{2mi} \left(\psi^* \frac{d\psi}{dx} - \frac{d\psi^*}{dx} \psi \right)$$

(d) If two operators commute, what can be said about their eigenvectors and eigenvalues?

One can find a complete set of states that are eigenvectors of both operators. The eigenvalues do not have to be the same.

2. **One Dimensional Schrodinger Equation:**
Consider the one dimensional potential $V(x) = \alpha\delta(x)$ with $\alpha < 0$. In the following take the energy to be less than zero, $E < 0$.

(a) What is the general form of the solutions to the time independent Schrodinger equation for $x < 0$? Eliminate any unphysical terms.

$A e^{\rho x}$

$$\rho = \sqrt{\frac{2m(-E)}{\hbar^2}} = \sqrt{\frac{2m|E|}{\hbar^2}}$$

(b) What is the general form of the solutions to the time independent Schrodinger equation for $x > 0$? Eliminate any unphysical terms.

$B e^{-\rho x}$ ρ is the same.

(c) What are the boundary conditions at $x = 0$?

Continuity at $x=0$: $A = B$

Delta fnt. boundary condition: $(-\rho - \rho) A = \frac{2m\alpha}{\hbar^2} A$

(d) Solve for the bound state energy or energies (if more than one).

$$-\cancel{2}\rho = \frac{\cancel{2}m\alpha}{\hbar^2} \rightarrow \rho^2 = \frac{2m(-E)}{\hbar^2} = \frac{m^2\alpha^2}{\hbar^4}$$

$$\rightarrow \boxed{E = -\frac{1}{2}\frac{m\alpha^2}{\hbar^2}}$$

3. **Harmonic Oscillator:**

The state of a one dimensional harmonic oscillator at $t = 0$ is

$$\psi(x,0) = \frac{\sqrt{3}}{2}\psi_1(x) + \frac{i}{2}\psi_2(x),$$

where $\psi_n(x)$ for $n = 0, 1, 2, \ldots$ are the harmonic oscillator energy eigenstates.

(a) What is the wave function at time t?

$$\psi(x,t) = \frac{\sqrt{3}}{2} e^{-i\frac{3}{2}\omega t}\psi_1(x) + \frac{i}{2} e^{-i\frac{5}{2}\omega t}\psi_2(x)$$

(b) What is the expectation value of x at time t?

$$\langle x \rangle = \int \left(\frac{\sqrt{3}}{2} e^{i\frac{3}{2}\omega t}\psi_1(x) - \frac{i}{2} e^{i\frac{5}{2}\omega t}\psi_2(x)\right)$$

$$\sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)\left(\frac{\sqrt{3}}{2} e^{-i\frac{3}{2}\omega t}\psi_1(x) + \frac{i}{2} e^{-i\frac{5}{2}\omega t}\psi_2(x)\right) dx$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\frac{\sqrt{3}}{2}\frac{1}{2}\left(-i\sqrt{2}\, e^{i\omega t} + i\sqrt{2}\, e^{-i\omega t}\right)$$

$$\boxed{\langle x \rangle = \frac{\sqrt{3}}{2}\sqrt{\frac{\hbar}{m\omega}}\sin(\omega t)}$$

(c) If an energy measurement is made at time t, what are the possible outcomes and their associated probabilities?

Outcome	Probability	
$\frac{3}{2}\hbar\omega$	$\left\lvert \frac{\sqrt{3}}{2} e^{-i\frac{3}{2}\omega t} \right\rvert^2 = \frac{3}{4}$	75%
$\frac{5}{2}\hbar\omega$	$\left\lvert \frac{i}{2} e^{-i\frac{5}{2}\omega t} \right\rvert^2 = \frac{1}{4}$	25%

(d) What is the expectation value of the energy at time t?

$$\langle E \rangle = \frac{3}{4} \cdot \frac{3}{2}\hbar\omega + \frac{1}{4} \cdot \frac{5}{2}\hbar\omega$$

$$= \frac{14}{8}\hbar\omega = \frac{7}{4}\hbar\omega$$

4. **Formalism:**

The Hamiltonian of a two state system is

$$H = E_o \sigma_y = E_o \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

(a) What are the eigenvalues and eigenvectors of this Hamiltonian?

$E_0 \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$

$-E_0 \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

(b) The wave function at $t = 0$ is

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

What is the wave function at time t?

$$|\psi(t)\rangle = |E_0\rangle\langle E_0|\psi(0)\rangle e^{-iE_0t} + |-E_0\rangle\langle -E_0|\psi(0)\rangle e^{iE_0t}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \frac{1}{\sqrt{2}} e^{-iE_0t} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \frac{1}{\sqrt{2}} e^{iE_0t}$$

$$= \begin{pmatrix} \cos(E_0t/\hbar) \\ \sin(E_0t/\hbar) \end{pmatrix}$$

(c) At time t compute the expectation values of

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\langle \sigma_x \rangle = \left(\cos\left(\frac{E_0 t}{\hbar}\right) \ \sin\left(\frac{E_0 t}{\hbar}\right) \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(E_0 t/\hbar) \\ \sin(E_0 t/\hbar) \end{pmatrix}$$

$$= 2 \sin(E_0 t/\hbar) \cos(E_0 t/\hbar) = \underline{\underline{\sin(2E_0 t/\hbar)}}$$

$$\langle \sigma_z \rangle = \left(\cos\left(\frac{E_0 t}{\hbar}\right) \ \sin\left(\frac{E_0 t}{\hbar}\right) \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(E_0 t/\hbar) \\ \sin(E_0 t/\hbar) \end{pmatrix}$$

$$= \cos^2\left(\frac{E_0 t}{\hbar}\right) - \sin^2\left(\frac{E_0 t}{\hbar}\right) = \underline{\underline{\cos(2E_0 t/\hbar)}}$$

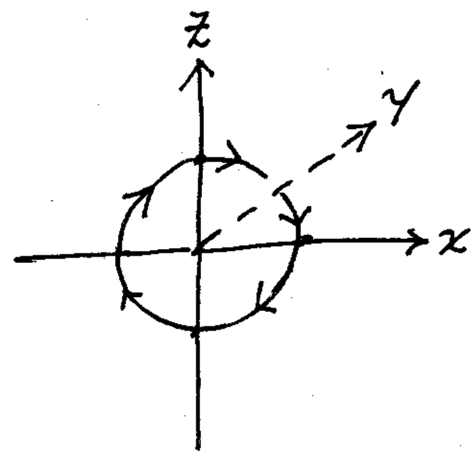
(d) Interpret your results in part (c) in terms of a spin precessing in a magnetic field. Sketch the path of the spin in three dimensions.

$t = 0$ $\quad \langle \sigma_z \rangle = 1 \quad , \langle \sigma_x \rangle = 0$

$\frac{2E_0 t}{\hbar} = \frac{\pi}{2}$ $\quad \langle \sigma_z \rangle = 0 \quad , \langle \sigma_x \rangle = 1$

$\frac{2E_0 t}{\hbar} = \pi$ $\quad \langle \sigma_z \rangle = -1 \quad , \langle \sigma_x \rangle = 0$

$\frac{2E_0 t}{\hbar} = \frac{3\pi}{2}$ $\quad \langle \sigma_z \rangle = 0 \quad , \langle \sigma_x \rangle = -1$



Precesses about $\hat{y}$ axis, which is the direction of the effective magnetic field.

5. **Angular momentum:**

(a) Compute the commutator $[L_z, L_x L_y]$.

$$= [L_z, L_x] L_y + L_x [L_z, L_y]$$

$$= i\hbar (L_y^2 - L_x^2)$$

(b) What is the matrix element $\langle \frac{3}{2}, \frac{3}{2} | J_x J_y | \frac{3}{2}, \frac{3}{2} \rangle$?

$$J_x = \frac{J_+ J_-}{2}, \quad J_y = \frac{J_+ - J_-}{2i}$$

The non-zero contribution comes from:

$$\langle \tfrac{3}{2}, \tfrac{3}{2} | \frac{-1}{4i} J_+ J_- | \tfrac{3}{2}, \tfrac{3}{2} \rangle$$

$$= -\frac{\hbar^2}{4i} \underbrace{\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)}}_{\text{from } J_+ \text{ on } |\frac{3}{2}, \frac{1}{2}\rangle} \underbrace{\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1) - \tfrac{3}{2}(\tfrac{3}{2}-1)}}_{\text{from } J_- \text{ on } |\frac{3}{2}, \frac{3}{2}\rangle}$$

$$= -\frac{\hbar^2}{4i} \cdot \frac{15-3}{4} = \frac{-3\hbar^2}{4i} = \frac{3i}{4}\hbar^2$$

(c) In adding $j_1 = 2$ and $j_2 = 1/2$ angular momentum, what is the state with $j = 5/2$ and $m = 3/2$? Hint: Start with the $j = 5/2$, $m = 5/2$ state and use the lowering operator.

$$J_-|5/2, 5/2\rangle = \hbar\underbrace{\sqrt{\tfrac{5}{2}\left(\tfrac{5}{2}+1\right) - \tfrac{5}{2}\left(\tfrac{5}{2}-1\right)}}_{\sqrt{5}}\,\left|\tfrac{5}{2}, \tfrac{3}{2}\right\rangle$$

$$= J_-|2,2\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \hbar\underbrace{\sqrt{2(2+1)-2(2-1)}}_{2}\,|2,1\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

$$+ \hbar\underbrace{\sqrt{\tfrac{1}{2}\left(\tfrac{1}{2}+1\right) - \tfrac{1}{2}\left(\tfrac{1}{2}-1\right)}}_{1}\,|2,2\rangle\left|\tfrac{1}{2},\tfrac{-1}{2}\right\rangle$$

$$\rightarrow \left|\tfrac{5}{2},\tfrac{3}{2}\right\rangle = \frac{2}{\sqrt{5}}|2,1\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \frac{1}{\sqrt{5}}|2,2\rangle\left|\tfrac{1}{2},\tfrac{-1}{2}\right\rangle$$

(d) A $J_z^{(1)}$ measurement is performed on the $|j = 5/2, m = 3/2\rangle$ state. $J_z^{(1)}$ acts only on particle one, which has $j_1 = 2$. What are the possible outcomes and their associated probabilities?

Outcome	Probability	
$\hbar$	$\left\lvert\frac{2}{\sqrt{5}}\right\rvert^2 = \frac{4}{5}$	80%
$2\hbar$	$\left\lvert\frac{1}{\sqrt{5}}\right\rvert^2 = \frac{1}{5}$	20%

6. **Bonus:** Do not spend time on this unless you are sure you have finished the rest of the exam.

(a) Consider 10 spin 1/2 particles. How many linearly independent states can you make with a total spin of 0? (This is adding angular momentum except with 10 spins instead of 2.)

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