"Analytic Method" to solve harmonic oscillator differential equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi$$

Step 1: Make this dimensionless:

$$\underbrace{-\frac{d^2\psi}{dx^2}}_{\frac{1}{L^2}[\psi]}+\underbrace{\frac{m^2\omega^2}{\hbar^2}x^2\psi}_{\left[\frac{m^2\omega^2}{\hbar^2}\right]L^2[\psi]}=\frac{2mE}{\hbar^2}\psi$$

$$\rightarrow\left[\frac{m^2\omega^2}{\hbar^2}\right]=\frac{1}{L^4}\rightarrow\left[\sqrt{\frac{\hbar}{m\omega}}\right]=L\Rightarrow$$ characteristic length scale is $\sqrt{\frac{\hbar}{m\omega}}$

Let $\xi=\dfrac{x}{\sqrt{\frac{\hbar}{m\omega}}}$.

Multiply by $\left(\sqrt{\frac{\hbar}{m\omega}}\right)^2$:

$$-\frac{d^2\psi}{d\xi^2}+\xi^2\psi=\underbrace{\frac{2E}{\hbar\omega}}_{\substack{\text{Dimensionless,}\\ \equiv K}}\psi$$

The characteristic energy is $\hbar\omega$ (or $\frac{\hbar\omega}{2}$).

Step 2: Asymptotics

$$\boxed{\frac{d^2\psi}{d\xi^2} = (\xi^2 - K)\psi} \quad \dots (1)$$

For large x or ξ this becomes

$$\frac{d^2\psi}{d\xi^2} = \xi^2\psi,$$

which has solution $Ae^{-\xi^2/2} + \underbrace{Be^{\xi^2/2}}_{\text{not normalizable}}$.

→ Write $\boxed{\psi(\xi) = h(\xi)e^{-\xi^2/2}}$ $\dots (2)$

Step 3: Series solution

Substitute back in: (2) → (1)

$$\frac{d^2h}{d\xi^2} - 2\xi\frac{dh}{d\xi} + (K-1)h = 0$$

$$h(\xi) = a_0 + a_1\xi + a_2\xi^2 + \dots = \sum_{j=0}^{\infty} a_j\xi^j$$

$$\frac{dh}{d\xi} = a_1 + 2a_2\xi + 3a_3\xi^2 + \dots = \sum_{j=0}^{\infty} j\,a_j\xi^{j-1}$$

$$\frac{d^2h}{d\xi^2} = 2a_2 + 2\cdot 3\cdot a_3\xi + \dots = \sum_{j=0}^{\infty}(j+1)(j+2)a_{j+2}\xi^j$$

$$\sum_{j=0}^{\infty}\left[(j+1)(j+2)a_{j+2} - 2ja_j + (K-1)a_j\right]\xi^j = 0$$

$$\rightarrow (j+1)(j+2)\,a_{j+2} - 2ja_j + (K-1)a_j = 0$$

$$\rightarrow \boxed{a_{j+2} = \frac{(2j+1-K)}{(j+1)(j+2)}\,a_j}$$ Recursion formula

The even & odd coefficients are decoupled:

even: $a_2 = \frac{(1-K)}{2}a_0,\ a_4 = \frac{(5-K)}{12}a_2 = \frac{(5-K)(1-K)}{24}a_0, \ldots$

odd: $a_3 = \frac{(3-K)}{6}a_1,\ a_5 = \frac{(7-K)}{20}a_3 = \frac{(7-K)(3-K)}{120}a\ , \ldots$

$$\rightarrow h = h_{even}(\xi) + h_{odd}(\xi)$$

determined by a_0 (and K) — determined by a_1 (and K)

For both cases for large j:

$$a_{j+2} \approx \frac{2}{j}a_j\ .$$

The series for e^{ξ^2} is

$$e^{\xi^2} = \sum_{j=0,2,4,\dots}^{\infty} \frac{1}{(j/2)!} \xi^j .$$

→ For this series we also have $\dfrac{a_{j+2}}{a_j} = \dfrac{1}{\frac{j+2}{2}}$

$$\sim \frac{1}{j/2} .$$

→ Both series (even & odd) go as e^{ξ^2}

UNLESS the series terminates for some integer:

$$0 = a_{n+2} = \frac{(2n+1-K)}{(n+1)(n+2)} a_n$$

→ $K = 2n+1$ for $n = 0, 1, 2, \dots$

→ $E = \dfrac{\hbar\omega}{2} K = \hbar\omega\left(n+\frac{1}{2}\right)$ ✓

K	n	$h(\xi)$
1	0	a_0
3	1	$a_1 \xi$
5	2	$a_0(1-2\xi^2)$
7	3	$a_1(\xi - \frac{2}{3}\xi^3)$