Homework 6
(due Friday, October, 18)

The purpose of this homework assignment is to give you practice finding eigenvalues and eigenvectors and to become more familiar with the Dirac notation.

1. We begin with a two state system with states labeled by $|1\rangle$ and $|2\rangle$. This may seem unphysical; however, there are many two state systems in quantum mechanics such spin 1/2 particles. The Hamiltonian we consider is

$$H = E_0 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = E_0(i|2\rangle\langle 1| - i|1\rangle\langle 2|). \tag{1}$$

   (a) Compute the eigenvalues of H.

   (b) Compute the eigenvectors of H, normalize them, and express them both as column vectors and in terms of $|1\rangle$ and $|2\rangle$.

   (c) Denoting the two eigenvectors as $|\psi_a\rangle$ and $|\psi_b\rangle$, compute $|\psi_a\rangle\langle\psi_a|$ and $|\psi_b\rangle\langle\psi_b|$.

   (d) Verify that

$$\begin{aligned} |\psi_a\rangle\langle\psi_a| + |\psi_b\rangle\langle\psi_b| &= \mathbf{1} & (2) \\ \lambda_a|\psi_a\rangle\langle\psi_a| + \lambda_b|\psi_b\rangle\langle\psi_b| &= H, & (3) \end{aligned}$$

   where $\mathbf{1}$ is the identity matrix.

   (e) Suppose at $t = 0$, the system is in state 1: $|\psi(t = 0)\rangle = |1\rangle$. Express $|\psi(t = 0)\rangle$ in terms of $|\psi_a\rangle$ and $|\psi_b\rangle$ using

$$|\psi(0)\rangle = |\psi_a\rangle\langle\psi_a|\psi(0)\rangle + |\psi_b\rangle\langle\psi_b|\psi(0)\rangle. \tag{4}$$

   (f) Compute $|\psi(t)\rangle$ using

$$|\psi(t)\rangle = |\psi_a\rangle\langle\psi_a|\psi(0)\rangle e^{-iE_a t/\hbar} + |\psi_b\rangle\langle\psi_b|\psi(0)\rangle e^{-iE_b t/\hbar}. \tag{5}$$

   (g) What is the probability at time t that the system is in state $|1\rangle$? (Hint: compute $|\langle 1|\psi(t)\rangle|^2$.)

2. Now we consider a three state system and perform the same kinds of calculations on it.

$$H = E_0 \begin{pmatrix} 1 & i & -1 \\ -i & -1 & -i \\ -1 & i & 1 \end{pmatrix} \tag{6}$$

   This can be thought of three atoms in a triangle where an electron can sit on any of the three atomic states, labeled by $|1\rangle$, $|2\rangle$, $|3\rangle$.

   (a) Compute the eigenvalues of H.

   (b) Compute the eigenvectors of H, normalize them, and express them both as column vectors and in terms of $|1\rangle$, $|2\rangle$ and $|3\rangle$.

(c) Denoting the three eigenvectors as $|\psi_a\rangle$, $|\psi_b\rangle$, and $|\psi_c\rangle$, compute $|\psi_a\rangle\langle\psi_a|$, $|\psi_b\rangle\langle\psi_b|$, and $|\psi_c\rangle\langle\psi_c|$.

(d) Verify that

$$\begin{aligned} |\psi_a\rangle\langle\psi_a| + |\psi_b\rangle\langle\psi_b| + |\psi_c\rangle\langle\psi_c| &= \mathbf{1} \qquad (7) \\ \lambda_a|\psi_a\rangle\langle\psi_a| + \lambda_b|\psi_b\rangle\langle\psi_b| + \lambda_c|\psi_c\rangle\langle\psi_c| &= H, \qquad (8) \end{aligned}$$

where **1** is the identity matrix.

(e) Suppose at $t = 0$, the system is in state 1: $|\psi(t = 0)\rangle = |1\rangle$. Express $|\psi(t = 0)\rangle$ in terms of $|\psi_a\rangle$, $|\psi_b\rangle$, and $|\psi_c\rangle$ using

$$|\psi(0)\rangle = |\psi_a\rangle\langle\psi_a|\psi(0)\rangle + |\psi_b\rangle\langle\psi_b|\psi(0)\rangle + |\psi_c\rangle\langle\psi_c|\psi(0)\rangle. \qquad (9)$$

(f) Compute $|\psi(t)\rangle$ using

$$|\psi(t)\rangle = |\psi_a\rangle\langle\psi_a|\psi(0)\rangle e^{-iE_at/\hbar} + |\psi_b\rangle\langle\psi_b|\psi(0)\rangle e^{-iE_bt/\hbar} + |\psi_c\rangle\langle\psi_c|\psi(0)\rangle e^{-iE_ct/\hbar}. \qquad (10)$$

(g) What is the probability at time t that the system is in state $|1\rangle$? (Hint: compute $|\langle 1|\psi(t)\rangle|^2$.)