Path Integrals in 1D

Lagrangian formulation of mechanics:

$$\mathcal{L} = \frac{1}{2} m (\dot{x})^2 - U(x)$$

$$S = \int_{t_i}^{t_f} \mathcal{L}\, dt = \int_{t_i}^{t_f} \left(\frac{1}{2} m (\dot{x}(t))^2 - U(x(t))\right) dt$$

Look for extrema: $x(t) + \delta x(t)$.

$$S = \int_{t_i}^{t_f} \left(\frac{1}{2} m (\dot{x} + \delta\dot{x})^2 - U(x + \delta x)\right) dt$$

$$\approx S|_{\delta x = 0} + \int_{t_i}^{t_f} \left(m \dot{x}\, \delta\dot{x} - \frac{dU}{dx}\, \delta x\right) dt$$

$$= S|_{\delta x = 0} + \int_{t_i}^{t_f} \left(-m\ddot{x} - \frac{dU}{dx}\right) \delta x\, dt$$

... note $\delta x(t_i) = 0$ & $\delta x(t_f) = 0$

$$\rightarrow \quad -m\ddot{x} - \frac{dU}{dx} = 0 \rightarrow m\ddot{x} = -\frac{dU}{dx}, \text{ i.e. } F = ma.$$

Time evolution operator

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar} |\varphi_n\rangle$$

$$|\psi(0)\rangle = \sum_n c_n |\varphi_n\rangle$$

$$|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle$$

Real space propagator

$$|\psi(0)\rangle = |x_i\rangle$$

$$\langle x_f | \underbrace{e^{-iHt} |x_i\rangle}_{|\psi(t)\rangle}$$

Amplitude for being at $|x_f\rangle$ at time t.

Break up t interval into N parts

$$\Delta t = t/N$$

$t_0 = 0 \quad t_1 = \Delta t \quad t_2 = 2\Delta t \quad t_N = N\Delta t = t$

$$\langle x_f | e^{-i\frac{Ht}{\hbar}} |x_i\rangle = \langle x_f | \underbrace{e^{-iH\Delta t/\hbar} e^{-iH\Delta t/\hbar} \cdots e^{-iH\Delta t/\hbar}}_{N \text{ terms}} |x_i\rangle$$

Insert complete sets of states: $\mathbb{1} = \int dx\, |x\rangle\langle x|$

$$\langle x_f | e^{-i\frac{Ht}{\hbar}} | x_i \rangle = \int dx_1 \int dx_2 \cdots \int dx_{N-1}$$

$$\langle x_f | e^{-iH(t_N - t_{N-1})/\hbar} | x_{N-1} \rangle$$
$$\langle x_{N-1} | e^{-iH(t_{N-1} - t_{N-2})/\hbar} | x_{N-2} \rangle$$
$$\vdots$$
$$\langle x_1 | e^{-iH(t_1 - t_0)/\hbar} | x_i \rangle$$

Free particle case ($U = 0$):

$$\langle x' | e^{-iH\Delta t/\hbar} | x \rangle = \int \frac{dk}{2\pi} \underbrace{\langle x' | k \rangle}_{e^{ikx'}} e^{-iE_k \Delta t/\hbar} \underbrace{\langle k | x \rangle}_{e^{-ikx}}$$

$$= \int \frac{dk}{2\pi} \exp\left(-i\left(\frac{\hbar k^2}{2m}\Delta t + k(x - x')\right)\right)$$

$$= \int \frac{dk}{2\pi} \exp\left(-i\frac{\hbar \Delta t}{2m}\left(k^2 + \frac{2mk}{\hbar \Delta t}(x - x')\right)\right)$$

complete the square

$$= \int \frac{dk}{2\pi} \exp\left(-i\frac{\hbar\Delta t}{2m}\left(k + \frac{m(x-x')}{\hbar \Delta t}\right)^2 + \frac{i\hbar\Delta t}{2m}\frac{m^2(x-x')^2}{\hbar^2 \Delta t^2}\right)$$

$$= \underbrace{\frac{1}{2\pi}\frac{\sqrt{\pi}}{\sqrt{i\frac{\hbar\Delta t}{2m}}}}_{\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{1/2}} \exp\left(\frac{i}{\hbar}\frac{1}{2}m\frac{(x-x')^2}{\Delta t}\right)$$

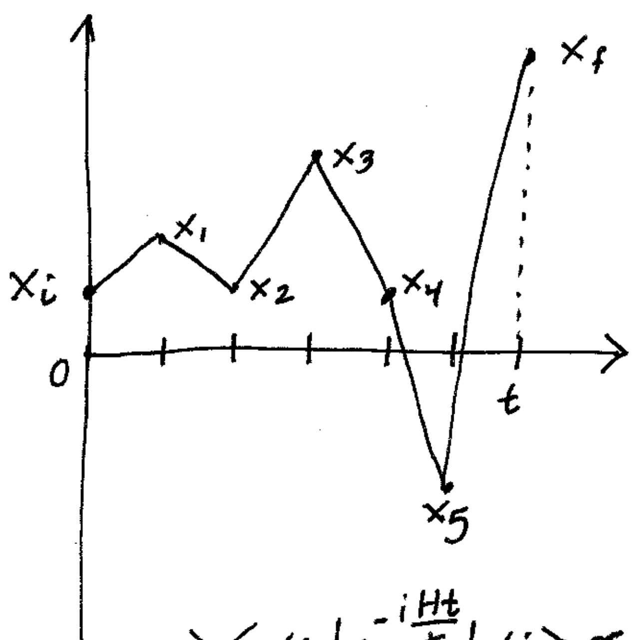
with $\sqrt{i} = \frac{1+i}{\sqrt{2}}$

$$\langle x_f | e^{-i\frac{Ht}{\hbar}} | x_i \rangle = \overbrace{\left(\frac{m}{2\pi i \hbar \Delta t}\right)^{N/2} \int dx_1 \int dx_2 \cdots \int dx_{N-1}}^{\text{Define as } \int \mathcal{D}x} \times$$

$$\times \underbrace{\exp\left(\frac{i}{\hbar}\left\{\frac{1}{2} m \frac{(x_N - x_{N-1})^2}{(\Delta t)^2} + \frac{1}{2} m \frac{(x_{N-1} - x_{N-2})^2}{(\Delta t)^2} + \cdots + \frac{1}{2} m \frac{(x_1 - x_0)^2}{(\Delta t)^2}\right\} \Delta t\right)}_{\approx \exp\left(\frac{i}{\hbar} S[x]\right)},$$

where $S[x] = \int_0^t dt' \left[\frac{1}{2} m \dot{x}^2\right]$ when $V=0$

$$\rightarrow \boxed{\langle x_f | e^{-i\frac{Ht}{\hbar}} | x_i \rangle = \int \mathcal{D}x \, e^{\frac{i}{\hbar} S[x]}}$$



Nearmost paths $e^{\frac{i}{\hbar} S[x]}$ oscillates rapidly & tends to cancel out; however, for the extrema (classical path) oscillate slowly.

$$\rightarrow \langle x_f | e^{-i\frac{Ht}{\hbar}} | x_i \rangle \propto e^{\frac{i}{\hbar} S_{cl}[x]}$$

Include V

$$e^{-iH\Delta t/\hbar} = e^{-\frac{i}{\hbar}\left(\frac{p^2}{2m}+V\right)\Delta t}$$

$$= e^{-\frac{i}{\hbar}\frac{p^2}{2m}\Delta t}\, e^{-\frac{i}{\hbar}V\Delta t} + \underbrace{\mathcal{O}(\Delta t)^2}_{\left(\frac{t}{N}\right)^2}$$

There are N such terms $\rightarrow \mathcal{O}\left(\frac{1}{N}\right) \rightarrow 0$ as $N \rightarrow \infty$.

Now in the exponential we have

$$\exp\left(\frac{i}{\hbar}\left\{\frac{1}{2}\frac{m(x_N - x_{N-1})^2}{(\Delta t)^2} - V(x_{N-1}) + \frac{1}{2}\frac{m(x_{N-1}-x_{N-2})^2}{(\Delta t)^2} - V(x_{N-2})\right.\right.$$

$$\left.\left. + \cdots + \frac{1}{2}\frac{m(x_1-x_0)^2}{(\Delta t)^2} - V(x_0)\right\}\Delta t\right)$$

$$\approx \exp\left(\frac{i}{\hbar}S[x]\right) \text{ with } \boxed{S[x] = \int_0^t dt' \underbrace{\left[\frac{1}{2}m\dot{x}^2 - V(x)\right]}_{\mathcal{L}}}$$