Uncertainty Principle:

Suppose A & B are two Hermitian operators, and $|\psi\rangle$ is an eigenvector of both of them:

$$A|\psi\rangle = a|\psi\rangle \quad , \quad B|\psi\rangle = b|\psi\rangle.$$

Since they are Hermitian, a & b are real.

$$\rightarrow \quad AB|\psi\rangle = Ab|\psi\rangle = bA|\psi\rangle = ba|\psi\rangle$$
$$BA|\psi\rangle = Ba|\psi\rangle = aB|\psi\rangle = ab|\psi\rangle$$

equal

$$\rightarrow \quad [A,B]|\psi\rangle = 0$$

$$\boxed{\text{If there is a complete set of simultaneous eigenvectors } (|\psi\rangle) \text{ of } A \,\&\, B\text{, then } [A,B]=0.}$$

What about the reverse?

$$[A,B]=0 \quad \rightarrow \text{ simultaneous eigenvalues}$$

Suppose $[A, B] = 0$. Let $|\psi_n\rangle$ be a complete set of eigenvectors of A:

$$A|\psi_n\rangle = a_n|\psi_n\rangle$$

For starters suppose the a_n are unique. (non-degenerate case)

$\rightarrow \langle\psi_m|[A,B]|\psi_n\rangle = 0$

$\rightarrow \langle\psi_m|AB|\psi_n\rangle = \langle\psi_m|BA|\psi_n\rangle$

$a_m\langle\psi_m|B|\psi_n\rangle = a_n\langle\psi_m|B|\psi_n\rangle$

$\rightarrow$ For $m \neq n$, $\langle\psi_m|B|\psi_n\rangle = 0$.

no off diagonal matrix elements

In matrix form

$$A = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & a_3 & \\ 0 & & & \ddots \end{pmatrix} \qquad B = \begin{pmatrix} b_1 & & & 0 \\ & b_2 & & \\ & & b_3 & \\ 0 & & & \ddots \end{pmatrix}$$

eigenvalues

$\rightarrow$ The eigenvectors of A are also eigenvectors of B.

Non-degenerate case: For simplicity take $a_1 = a_2$, but all the other eigenvalues are distinct. Now we have

$$a_1 \langle \psi_1 | B | \psi_2 \rangle = a_1 \langle \psi_1 | B | \psi_2 \rangle$$

& in general $\langle \psi_1 | B | \psi_2 \rangle \neq 0$.

$$A = \begin{pmatrix} a_1 & & & \\ & a_1 & & \\ & & a_3 & \\ & & & \ddots \end{pmatrix} \qquad B = \left(\begin{array}{cc|cc} b_1 & b' & & \\ b'^* & b_2 & & 0 \\ \hline & & b_3 & \\ & 0 & & \ddots \end{array}\right)$$

$|\psi_1\rangle$ & $|\psi_2\rangle$ are not necessarily eigenvectors of B, but by diagonalizing $\begin{pmatrix} b_1 & b' \\ b'^* & b_2 \end{pmatrix}$,

we can find eigenvectors of B of the form

$$\alpha |\psi_1\rangle + \beta |\psi_2\rangle,$$

which also has $A(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) = a_1(\alpha|\psi_1\rangle + \beta|\psi_2\rangle)$. In other words we have simultaneous eigenvector of A & B. The general non-degenerate case follows in a similar manner.

$$\boxed{[A, B] = 0 \iff \text{complete set simultaneous eigenvectors}}$$

What happens if $[A,B] \neq 0$?

Uncertainty principle: $\boxed{\sigma_A \sigma_B \geq \left| \frac{1}{2i} \langle [A,B] \rangle \right|}$

Note if $A|\psi\rangle = a|\psi\rangle$, then

$A^2|\psi\rangle = a^2|\psi\rangle$ and

$\sigma_A^{\,2} \quad = (a^2 - a^2) = 0.$

Proof:

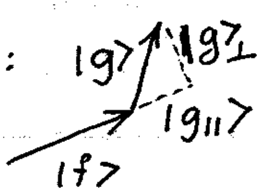
$$\sigma_A^2 = \langle\psi|(A - \langle A\rangle)(A - \langle A\rangle)|\psi\rangle \equiv \langle f|f\rangle,$$
$$\sigma_B^2 = \langle\psi|(B - \langle B\rangle)(B - \langle B\rangle)|\psi\rangle \equiv \langle g|g\rangle,$$

where $|f\rangle = (A - \langle A\rangle)|\psi\rangle$

$|g\rangle = (B - \langle B\rangle)|\psi\rangle.$

Schwarz inequality: $\boxed{\langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2}$

Picture:



$|g_{\parallel}\rangle = |f\rangle \dfrac{\langle f|g\rangle}{\langle f|f\rangle}$

$|g_\perp\rangle = |g\rangle - |g_{\parallel}\rangle$

$\rightarrow \langle f|g_{\parallel}\rangle = \langle f|g\rangle$

$\langle f|g_\perp\rangle = \langle f|g\rangle - \langle f|g\rangle = 0$

$\langle g_{\parallel}|g_\perp\rangle = \dfrac{\langle f|g\rangle^*}{\langle f|f\rangle} \langle f|g_\perp\rangle = 0$

$$\rightarrow \langle g|g\rangle = \langle g_{\parallel}|g_{\parallel}\rangle + \langle g_{\perp}|g_{\perp}\rangle$$

$$\geqslant \langle g_{\parallel}|g_{\parallel}\rangle = \frac{|\langle f|g\rangle|^2}{\langle f|f\rangle}$$

$$\rightarrow \langle f|f\rangle\langle g|g\rangle \geqslant |\langle f|g\rangle|^2 \checkmark$$

We now have

$$\sigma_A^2\sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \geqslant |\langle f|g\rangle|^2$$

$$\geqslant (\mathrm{Im}(\langle f|g\rangle))^2,$$

where $\mathrm{Im}(\langle f|g\rangle) = \dfrac{\langle f|g\rangle - \langle f|g\rangle^*}{2i}$

$$= \frac{\langle f|g\rangle - \langle g|f\rangle}{2i}$$

$$\begin{array}{rl} & \langle f|g\rangle = \langle\psi|(A-\langle A\rangle)(B-\langle B\rangle)|\psi\rangle \\ - & \underline{\langle g|f\rangle = \langle\psi|(B-\langle B\rangle)(A-\langle A\rangle)|\psi\rangle} \\ & \langle\psi|[A,B]|\psi\rangle \end{array}$$

$$\rightarrow \sigma_A^2\sigma_B^2 \geqslant \left(\frac{\langle[A,B]\rangle}{2i}\right)^2 \checkmark$$