Proof of the Uncertainty Principle

Introduction

This is a simplified proof of the uncertainty principle. We will do a more general proof later, but I think it is useful to do a proof of a special case now if the proof is transparent. At the end of this document I show how this special case can be generalized to include all wave functions.

Special Case

The special case we consider is that the expectation value of the position and momentum are zero. In this case the uncertainty principle reduces to

$$\begin{aligned}
(\Delta x)^2 &= \langle x^2\rangle - \langle x\rangle^2 = \langle x^2\rangle = \int dx\ \psi^* x^2 \psi = \int dx\ (x\psi)(x\psi) \\
(\Delta p)^2 &= \langle p^2\rangle - \langle p\rangle^2 = \langle p^2\rangle = \int dx\ \psi^* \left(-\hbar^2 \frac{d^2}{dx^2}\right)\psi = \hbar^2 \int dx \left(\frac{d\psi}{dx}\right)\left(\frac{d\psi}{dx}\right) \\
\Delta x \Delta p &\geq \frac{\hbar}{2}.
\end{aligned}$$

Cauchy-Schwarz Inequality

You are familiar with this inequality from three dimensional vectors. If $\vec{A}$ and $\vec{B}$ are two three dimensional vectors, then $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos(\theta)$ and in particular

$$|\vec{A}\cdot\vec{B}| \leq |\vec{A}||\vec{B}|. \tag{1}$$

This is the Cauchy-Schwarz inequality. Let us prove it in a way that generalizes to the case at hand.

The basic idea is that there is a part of $\vec{A}$ which is in the direction of $\vec{B}$ and a part of $\vec{A}$ which is perpendicular to $\vec{B}$. Thus, we can break $\vec{A}$ up into two vectors:

$$\vec{A} = \vec{A}_{\parallel} + \vec{A}_{\perp} \tag{2}$$

$$\vec{B}\cdot\vec{A}_{\perp} = 0. \tag{3}$$

The parallel and perpendicular vectors are given by

$$\vec{A}_{\parallel} = \frac{\vec{B}}{|\vec{B}|}\left(\frac{\vec{A}\cdot\vec{B}}{|\vec{B}|}\right) \tag{4}$$

$$\vec{A}_{\perp} = \vec{A} - \vec{A}_{\parallel}. \tag{5}$$

Now because $\vec{A}_{\parallel}$ and $\vec{A}_{\perp}$ are orthogonal

$$|\vec{A}|^2 = \vec{A}_{\parallel}\cdot\vec{A}_{\parallel} + \vec{A}_{\perp}\cdot\vec{A}_{\perp}. \tag{6}$$

This is basically the Pythagorean theorem. The length of vector $\vec{A}$ is greater than or equal to the length of vector $\vec{A}_{\parallel}$.

$$|\vec{A}|^2 \geq \vec{A}_{\parallel}\cdot\vec{A}_{\parallel} = \frac{(\vec{A}\cdot\vec{B})^2}{|\vec{B}|^2}. \tag{7}$$

This is the Cauchy-Schwarz inequality:

$$|\vec{A}|^2|\vec{B}|^2 \geq (\vec{A}\cdot\vec{B})^2. \tag{8}$$

Cauchy-Schwarz inequality for functions

We will cover the results of this section rigorously in approximately a month. Thus, if this does not live up to your level of rigor, just wait until then. Consider two functions: $f(x)$ and $g(x)$. We can define a kind of dot product for these functions as follows

$$\text{“}f\cdot f\text{”} = \int dx\ f^*(x)f(x) \tag{9}$$

$$\text{“}g\cdot g\text{”} = \int dx\ g^*(x)g(x) \tag{10}$$

$$\text{“}f\cdot g\text{”} = \int dx\ f^*(x)g(x). \tag{11}$$

The complex conjugate is important here so that “$f\cdot f$” is greater than or equal to zero. Following a line of reasoning exactly the same as in the previous section we can prove the Cauchy-Schwarz identity in this case

$$\left|\int dx\ f^*(x)g(x)\right|^2 \leq \left(\int dx\ f^*(x)f(x)\right)\left(\int dx\ g^*(x)g(x)\right). \tag{12}$$

Uncertainty Principle

Let $f(x) = x\psi(x)$ and $g(x) = d\psi/dx$. Then

$$\int dx\ f^*(x)f(x) = \int dx\ |x\psi(x)|^2 = (\Delta x)^2 \tag{13}$$

$$\int dx\ g^*(x)g(x) = \int dx\ \left|\frac{d\psi}{dx}\right|^2 = \frac{(\Delta p)^2}{\hbar^2} \tag{14}$$

$$\int dx\ f^*(x)g(x) = \int dx\ (x\psi^*(x))\left(\frac{d\psi}{dx}\right) \tag{15}$$

$$\int dx\ g^*(x)f(x) = \int dx\ (x\psi(x))\left(\frac{d\psi^*}{dx}\right) \tag{16}$$

$$\mathrm{Re}\left\{\int dx\ f^*(x)g(x)\right\} = \int dx\ \frac{1}{2}\left(\psi^*\frac{d\psi}{dx} + \frac{d\psi}{dx}\psi^*\right) = \int dx\ x\frac{1}{2}\frac{d|\psi|^2}{dx} = -\frac{1}{2}. \tag{17}$$

Because

$$\left|\mathrm{Re}\left\{\int dx\ f^*(x)g(x)\right\}\right| \leq \left|\int dx\ f^*(x)g(x)\right|, \tag{18}$$

applying the Cauchy-Schwarz identity yields

$$\left(-\frac{1}{2}\right)^2 \leq (\Delta x)^2\frac{(\Delta p)^2}{\hbar^2}, \tag{19}$$

which is the uncertainty relation.

General Case

At the beginning of this document assumed that the expectation values of position and momentum were zero. Suppose that the expectation value of the position is x_o and the expectation value of the momentum is p_o. We can define a new wave function, $\tilde{\psi}$,

$$\tilde{\psi}(x) = e^{-ip_o x/\hbar}\psi(x + x_o), \tag{20}$$

which has zero expectation values for both position and momentum, and which also has the same Δx and Δp as ψ. Thus, we can without loss of generality assume that the expectation values of position and momentum are zero.