Vector space:

addition and scalar multiplication so that if v_1 & v_2 are vectors so is $c_1 v_1 + c_2 v_2$.

Associativity addition: $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$
Commutativity addition: $v_1 + v_2 = v_2 + v_1$
Zero vector: $0 + v = v + 0$ for any v
Inverse: $v + (-v) = (-v) + v = 0$
Distributivity scalar mult: $c(v_1 + v_2) = c v_1 + c v_2$
also $(c_1 + c_2) v = c_1 v + c_2 v$
and $c_1 (c_2 v) = (c_1 v_2) v$
Identity under scalar mult.: $1 v = v$

Wavefunctions, $\psi(x)$, form a vector space with the scalars being complex. The zero vector is the function which is zero everywhere.

In many cases we will denote the vectors as $|\psi\rangle$.

Inner product:

In 3D $\vec{v}\cdot\vec{v} = |\vec{v}|^2 \geq 0$
and $\vec{v}\cdot\vec{v} = 0$ iff $\vec{v} = \vec{0}$

For the function vector space the inner product is

$$\langle f | g \rangle = \int dx\, f^*(x)\, g(x).$$

Note that $\langle f|f \rangle = \int dx\, |f(x)|^2 \geq 0$
and $\int dx\, |f(x)|^2 = 0$ iff $f(x) = 0$.

Properties:

$$\langle f | g_1 + g_2 \rangle = \langle f | g_1 \rangle + \langle f | g_2 \rangle$$
$$\langle f_1 + f_2 | g \rangle = \langle f_1 | g \rangle + \langle f_2 | g \rangle$$
$$\langle f | c g \rangle = c \langle f | g \rangle$$
$$\langle c f | g \rangle = c^* \langle f | g \rangle$$
$$\langle f | g \rangle^* = \langle g | f \rangle$$

Basis:

In 2D $\vec{V} = c_1 \hat{x} + c_2 \hat{y}$, where

$$c_1 = \hat{x} \cdot \vec{V}$$
$$c_2 = \hat{y} \cdot \vec{V}$$

and $\hat{x} \cdot \hat{x} = 1$, $\hat{y} \cdot \hat{y} = 1$, $\hat{x} \cdot \hat{y} = 0$.

(orthonormal basis)

Consider either the eigenstates of the infinite square well ($0 < x < a$ and $n = 1, 2, 3, \ldots$) or the harmonic oscillator ($-\infty < x < \infty$ and $n = 0, 1, \ldots$)

ortho-normal

$$\left.\begin{aligned} \langle \psi_n | \psi_n \rangle &= \int dx\, \psi_n^*(x) \psi_n(x) = 1 \\ \langle \psi_n | \psi_m \rangle &= \int dx\, \psi_n^*(x) \psi_m(x) = 0 \text{ for } m \neq n \end{aligned}\right.$$

$$\left.\begin{aligned} \psi(x) &= \sum_n c_n \psi_n(x) \\ c_n &= \int dx\, \psi_n^*(x) \psi(x) \end{aligned}\right\} \text{completeness}$$

New notation:

$$|\psi\rangle = \sum_n c_n |\psi_n\rangle$$

$$c_n = \langle \psi_n | \psi \rangle$$

Just as a 2D vector $\vec{v} = c_1 \hat{x} + c_2 \hat{y}$ may be specified by its <u>components</u> c_1 & c_2 after a basis is chosen, we can specify vectors in the function space by the c_n.

Convention: write as a column vector

$$|\psi\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix}$$

It is also conventional to have a dual vector written as

$$\langle\psi| = \begin{pmatrix} c_1^* & c_2^* & c_3^* & \cdots \end{pmatrix}$$

Using matrix multiplication

$$\langle\psi|\psi\rangle = \begin{pmatrix} c_1^* & c_2^* & \cdots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$$

$$= |c_1|^2 + |c_2|^2 + \cdots = \int dx\, \psi^*(x)\psi(x)$$

Dirac $\underbrace{\text{bra}}_{\langle\psi|}\ \underbrace{\text{ket}}_{|\psi\rangle}$ notation.