NMR — Coax Cable, Impedance Matching

Experiment NA-CCIM

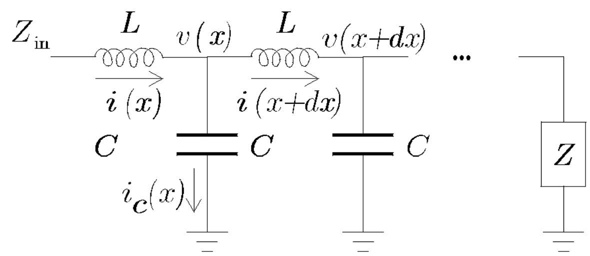

Figure 1: Model of a coaxial cable terminated in an arbitrary impedance Z.

Coaxial Cable

A lossless coax cable supports voltage and current waves traveling in both directions along the cable. The wave equations are easily derived as the limiting case of the circuit shown in Fig. 1 consisting of segments of parallel capacitance $C = c\,dx$ and series inductance $L = \ell\,dx$, where c and ℓ are the cable's capacitance and inductance per unit length.

The voltage drop across the inductor at any instant satisfies $v(x + dx, t) - v(x, t) = -L\,di/dt = -\ell dx\,di/dt$, giving

$$\frac{dv}{dx} = -\ell\frac{di}{dt} \tag{1}$$

while the current in the capacitor at any instant satisfies $i(x, t) - i(x + dx, t) = C\,dv/dt = c\,dx\,dv/dt$, giving

$$\frac{di}{dx} = -c\frac{dv}{dt} \tag{2}$$

Then for example, differentiating the first with respect to x and the second with respect to t and eliminating $d^2i/dxdt$ gives the wave equation for $v(x, t)$

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{v_0^2}\frac{\partial^2 v}{\partial t^2} = 0 \tag{3}$$

where

$$v_0^2 = \frac{1}{\ell c} \tag{4}$$

(Differentiation notation has been changed to the partial derivative symbol ∂ since v is a function of x and t.) In a similarly fashion it can be shown $i(x, t)$ also satisfies the wave equation.

$$\frac{\partial^2 i}{\partial x^2} - \frac{1}{v_0^2}\frac{\partial^2 i}{\partial t^2} = 0 \tag{5}$$

We assume an ideal voltage source drives one end of the cable (defined as $x = 0$) at an angular frequency ω and the other end of the cable (defined as $x = D$, where D is the cable length) is terminated by an impedance Z. The steady state solution to the wave equation for the voltage can then be expressed as the real part of a complex voltage

$$v(x, t) = v_1 e^{j(\omega t + kx)} + v_2 e^{j(\omega t - kx)} \tag{6}$$

where the wavenumber k and the frequency ω are linked by the dispersion relation

$$v_0 = \frac{\omega}{k} \tag{7}$$

Eq. 6 represents harmonic waves of complex amplitude v_1 and v_2 traveling in the negative and positive x directions. The cable characteristics ℓ and c and its length D as well as the value of the terminating load impedance Z will determine the relative magnitude and phases of v_1 and v_2, i.e., the complex ratio v_1/v_2 (also called the reflection coefficient). This ratio determines whether the voltage will exist as a pure standing wave (if $|v_1| = |v_2|$, i.e., $v_1/v_2 = e^{j\phi}$), or as a pure traveling wave propagating from source to load (if $v_1 = 0$, i.e., $v_1/v_2 = 0$), or a combination of a standing wave and a traveling wave (otherwise).

To see how this comes about from a phasor analysis, Eq. 6 is better expressed $v(x,t) = v(x)e^{j\omega t}$ where

$$v(x) = v_1 e^{jkx} + v_2 e^{-jkx} \tag{8}$$

Then Eq. 2 $di/dx = -c\,dv/dt = -j\omega c v(x) e^{j\omega t}$ gives $i(x,t) = i(x)e^{j\omega t}$ where $i(x)$ satisfies

$$\frac{di}{dx} = -j\omega c\left(v_1 e^{jkx} + v_2 e^{-jkx}\right) \tag{9}$$

and has the solution

$$\begin{aligned} i(x) &= -j\omega c\left(\frac{1}{jk} v_1 e^{jkx} - \frac{1}{jk} v_2 e^{-jkx}\right) \\ &= \frac{1}{Z_0}\left(-v_1 e^{jkx} + v_2 e^{-jkx}\right) \end{aligned} \tag{10}$$

where

$$Z_0 = \sqrt{\frac{\ell}{c}} \tag{11}$$

is a *real* constant of the cable called its characteristic impedance. Typical values for 50 Ω coax cable are $c = 1$ pf/cm, $\ell = 250$ pH/cm, $v_0 = 2 \times 10^{10}$ cm/s, and $Z_0 = 50\ \Omega$, only two of which can be considered independent parameters.

The ratio v_1/v_2 is obtained from the boundary condition at the load end of the cable $x = D$. At this end, the cable is terminated in some complex impedance Z giving

$$\begin{aligned} i(D) &= \frac{v(D)}{Z} \\ \frac{1}{Z_0}\left(-v_1 e^{j\delta} + v_2 e^{-j\delta}\right) &= \frac{1}{Z}\left(v_1 e^{j\delta} + v_2 e^{-j\delta}\right) \end{aligned} \tag{12}$$

where $\delta = kD$ now characterizes the cable length D. Solving for the required ratio

$$\frac{v_1}{v_2} = e^{-2j\delta}\left(\frac{Z - Z_0}{Z + Z_0}\right) \tag{13}$$

Equation 13 shows that a pure standing wave ($v_1/v_2 = e^{j\phi}$) will result if and only if Z is zero, infinite, or purely imaginary, i.e., a ground, an open, or a pure reactance. A pure traveling wave can result if and only if $Z = Z_0$, i.e., a pure resistive load equal to the cable's characteristic impedance.

The impedance at the front ($x = 0$) end of the cable $Z_{\rm in} = v(0)/i(0)$ becomes

$$\begin{aligned} Z_{\rm in} &= Z_0 \frac{v_1 + v_2}{-v_1 + v_2} \\ &= Z_0\left(\frac{1 + v_1/v_2}{1 - v_1/v_2}\right) \end{aligned} \tag{14}$$

Then usingsing it in Eq. 14 gives

$$Z_{\rm in} = Z_0 \frac{Z\cos\delta + jZ_0\sin\delta}{Z_0\cos\delta + jZ\sin\delta} \tag{15}$$

Eq. 15 can be used to demonstrate several special cases. For example, if $Z = Z_0$ (cable terminated in its characteristic impedance), $Z_{\rm in} = Z_0$ for any cable length. For $\delta = n\pi$ (cable length an integral number of half wavelengths), the impedance at the cable end is transferred perfectly to the input $Z_{\rm in} = Z$. For $\delta = (n + 1/2)\pi$ (cable odd number of 1/4 wavelengths), $Z_{\rm in} = Z_0^2/Z$. Such cables convert inductor behavior at their end, $Z = j\omega L$, to capacitor behavior at their input $Z_{\rm in} = 1/j\omega C_{\rm eff}$ with $C_{\rm eff} = L/Z_0^2$; or vice

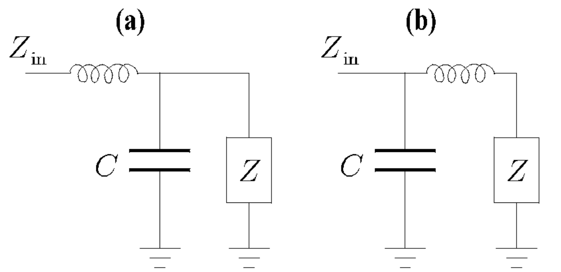

Figure 2: Cables short compared to a wavelength, can be treated as a series inductance and parallel capacitance in either order.

versa with $L_{\text{eff}} = C/Z_0^2$. They also convert short circuits $Z = 0$ to open circuits $Z_{\text{in}} = \infty$ and vice versa.

For short cables ($\delta \ll 1$), and to first order in δ Eq. 15 gives

$$Z_{\text{in}} = Z_0 \frac{Z + j\delta Z_0}{Z_0 + j\delta Z} \quad (16)$$

Noting that $\delta Z_0 = \omega \ell D = \omega L$ and $\delta / Z_0 = \omega c D = \omega C$ where C and L are the "total" cable capacitance and inductance (i.e., 1 pf and 250 pH per cm of cable), this can be rewritten

$$Z_{\text{in}} = \frac{Z + j\omega L}{1 + j\omega C Z} \quad (17)$$

Eq. 17 (again to first order in δ) can be treated as in either Fig. 2a or b. Of course, the approximation ($\delta = 2\pi D/\lambda$) $\ll 1$ should be checked. Note that for $v_0 = 2 \times 10^{10}$ cm/s, the product $\lambda f = 200$ MHz m, e.g., $\lambda = 20$ m at 10 MHz, and thus at this frequency $\delta = 0.1$ for a cable length $D = \pi/10 \approx 0.3$ m

Impedance Matching

Our NMR coil is placed at the end of a 40 cm coax cable for ease of placement between the poles of the high field magnet. While the 10 or so nH (40 cm cable) series inductance is negligible compared to the $\approx 1\,\mu$H NMR coil inductance, the cable's $C = 40$ pF parallel capacitance will not be negligible.

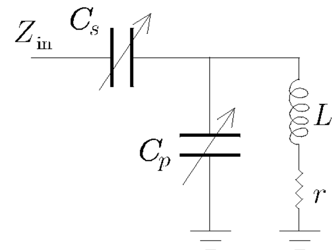

Figure 3: The tuning circuit for our NMR coil.

For impedance matching purposes, we will use a series-parallel resonance (tank) circuit as modeled in Fig. 3. The parallel capacitance to ground C_p will consists of the cable capacitance plus a discrete variable capacitor. The discrete series capacitance C_s is also variable. C_p and C_s must be adjusted to match the impedance of the tank circuit to 50Ω. That is, the real part of Z must be 50 Ω and the imaginary part must be zero.

The impedance Z of the series-parallel resonance circuit is

$$\begin{aligned} Z &= \frac{1}{j\omega C_s} + \frac{\frac{1}{j\omega C_p}(j\omega L + r)}{\frac{1}{j\omega C_p} + j\omega L + r} \\ &= \frac{1}{j\omega C_s} + \frac{j\omega L + r}{1 - \omega^2 L C_p + j\omega r C_p} \\ &= \frac{1 - \omega^2 L C_p + j\omega r C_p + j\omega C_s (j\omega L + r)}{j\omega C_s (1 - \omega^2 L C_p + j\omega r C_p)} \\ &= \frac{1 - \omega^2 L (C_s + C_p) + j\omega r (C_s + C_p)}{-\omega^2 r C_s C_p + j\omega C_s (1 - \omega^2 L C_p)} \quad (18) \end{aligned}$$

We need only solve for the real part of Z which neatly becomes

$$\Re\{Z\} = \frac{r}{(1 - \omega^2 L C_p)^2 + \omega^2 r^2 C_p^2} \quad (19)$$

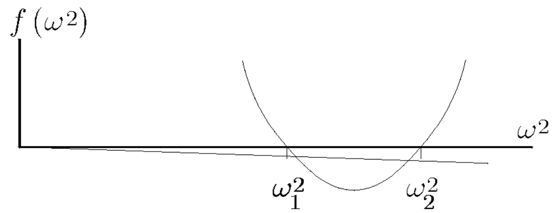

Figure 4: Graphical analysis of resonance equation. The parabolic curve is the left side of Eq. 20; the down-sloping straight line is the right side. The intersection points near ω_1^2 and ω_2^2 are the solutions.

Setting the imaginary part to zero yields the (resonance) equation

$$(1-\omega^2 LC_p)(1-\omega^2 L(C_s+C_p)) = -\omega^2 r^2 C_p(C_s+C_p) \tag{20}$$

The resonance equation has two roots (resonances). For $r = 0$, the resonance frequencies ω_1 and ω_2 satisfy $\omega_1^2 L(C_p + C_s) = 1$ and $\omega_2^2 LC_p = 1$.

The behavior (as a function of ω^2) of the left side of the resonance equation is quadratic and only negative in the region between the two $r = 0$ resonance frequencies as shown in Fig. 4. It reaches a minimum midway between ω_1^2 and ω_2^2 taking on the value $-C_s/4(C_s + C_p)$ there (around 0.05 for our apparatus). For finite r, the line representing the right side of the resonance equation can be expressed $-\omega^2/Q^2\omega_1\omega_2$ (where $Q^2 = \omega_1\omega_2 L^2/r^2$ is typically large) and between the resonances takes on values around $1/Q^2$ (about 0.001 for our apparatus). Thus, the two solutions for ω^2 at the intersection of the linear and quadratic terms remain near the $r = 0$ resonances.

If we take the frequency $\omega = 10^8$/s (f = 15 MHz) as a design parameter, and since C_p must be of the order of 100 pf (40 pf of cable capacitance plus the adjustable discrete capacitor of similar size), L will have to be designed around 1 μH. But once the coil is made, the capacitors will have to be adjusted for the actual coil inductance.

We need to look at the condition $\Re\{Z\} = 50\,\Omega$ to determine how to choose C_s and C_p in satisfying the resonance equation. Rewriting Eq. 19

$$\frac{r}{50\,\Omega} = (1-\omega^2 LC_p)^2 + \omega^2 r^2 C_p^2 \tag{21}$$

note that with $r \approx 1\,\Omega$, the left side is of order 10^{-1}-10^{-2}. The second term on the right side can be shown to be no bigger than $1/Q^2$ which is expected to be of the order $10^{-3} - 10^{-4}$ and thus can be neglected. Thus, the first term will have to be adjusted to meet the matching condition

$$\frac{r}{50\,\Omega} = (1-\omega^2 LC_p)^2 \tag{22}$$

If the resonance equation is satisfied at the first resonance: $\omega^2 L(C_s + C_p) \approx 1$, then $1 - \omega^2 LC_p \approx C_s/(C_s + C_p)$. Using this in Eq. 22 and solving for C_s gives

$$C_s \approx C_p\sqrt{\frac{r}{50\,\Omega}} \tag{23}$$

i.e., $C_s \approx 15$ pf.

If the resonance equation is satisfied at the second resonance: $\omega^2 \approx 1/LC_p$, then $1 - \omega^2 LC_p$ must be obtained from the resonance equation. The final result can be expressed

$$C_s \approx \frac{C_p}{Q^2\sqrt{r/50\,\Omega}} \tag{24}$$

and indicates a C_s smaller than expected stray capacitance would be needed.

Thus, the first resonance and impedance matching condition will be used.

The total current from the source i passes through the series capacitor and divides at the node between the parallel capacitor and the inductor-resistor combination $i = i_c + i_l$. The voltage at this node is the same for both paths and thus the capacitor current i_c and inductor current i_l satisfy $i_c Z_c = i_l Z_l$ where the Z's are each path's impedance. Together these two equations give

$$(i - i_l)\left(\frac{1}{j\omega C_p}\right) = i_l(j\omega L + r) \tag{25}$$

or

$$i_l = i\frac{1}{1 - \omega^2 L C_p + j\omega r C_p} \tag{26}$$

and the magnitude of $|i_l| = \sqrt{i_l^* i_l}$ gives

$$|i_l|^2 = \frac{|i|^2}{(1 - \omega^2 L C_p)^2 + \omega^2 r^2 C_p^2} \tag{27}$$

which from the impedance matching condition Eq. 21 gives

$$i_l^2 r = i^2 50\,\Omega \tag{28}$$

and demonstrates that the input power is completely delivered to the only real load r in the circuit.