Where from Lorentzian?
Addendum to SAS

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The Lorentzian derives from the equation of motion for the displacement x of a mass m subject to a linear restoring force -kx with a small amount of damping -$b\dot{x}$ and a harmonic driving force $F(t) = F_0\Re[e^{i\omega t}]$ set with an amplitude F_0 and driving frequency, i.e.,

$$m\ddot{x} + b\dot{x} + kx = F(t) \tag{1}$$

The restoring force constant $k = m\omega_0^2$ where ω_0 is the natural frequency of oscillation (free oscillation with no damping) and the constant b is a measure of the damping force -$b\dot{x}$ which is linear in the velocity. Dividing Eq. 1 by m gives

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = F(t)/m = f(t) = f_0\cos(\omega t) = f_0\Re[e^{i\omega t}] \tag{2}$$

where $\gamma = b/m$ is the damping constant in frequency units. The solution to Eq. 2 can be written in the form $x(t) = A\cos(\omega t + \delta) = A\Re[e^{i\omega t+\delta}]$ where the amplitude $A(\omega)$ and phase $\delta(\omega)$ are respectively written as

$$A(\omega) = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \tag{3}$$

and

$$\delta = \tan^{-1}\left(\frac{-\gamma\omega}{\omega_0^2 - \omega^2}\right) \tag{4}$$

Notice that the imaginary part of the phase is negative, implying that the phase angle is in the third and fourth quadrants with the displacement lagging the driving force.

The Lorentzian describes the line shape of transition processes involving atoms and laser fields. The three main processes are stimulated absorption, stimulated emission and spontaneous emission. The frequency dependence of these processes has a "Lorentzian"line shape characterized by the energy envelope of a lightly damped driven harmonic oscillator described by Eq. 1.

The total energy E of the oscillator is

$$E = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2 = \frac{1}{2}A^2[m\cos^2(\omega t + \delta) + m\omega^2\sin^2(\omega t + \delta)] \quad , \tag{5}$$

which after taking the time average and inserting Eq. 3 becomes

$$< E >= \frac{1}{4}A^2 m(\omega_0^2 + \omega^2) = \frac{1}{4}mf_0^2\frac{\omega_o^2 + \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \quad . \tag{6}$$

We now assume a small damping constant γ, or equivalently a high Q, with $< E >$ sharply peaked near ω_0 so that we can use the approximations $\omega \approx \omega_0$ and $\omega_0^2 - \omega^2 \approx 2\omega_0(\omega_0 - \omega)$ to obtain from Eq. 6 the result

$$< E >\approx \frac{1}{8}mf_0^2\frac{1}{4(\omega_0 - \omega)^2 + \gamma^2} \quad . \tag{7}$$

By making the substitutions $\nu = \omega/2\pi$ and $\nu_0 = \omega_0/2\pi$ for frequency in Hz and renormalizing the damping constant with the substitution $\Gamma = \gamma/2\pi$, Eq. 7 becomes

$$< E >= \frac{mf_0^2}{2(2\pi\Gamma)^2}\left(\frac{1}{1+4(\nu_0-\nu)^2)/\Gamma^2}\right) = \frac{mf_0^2}{2(2\pi\Gamma)^2}\mathcal{L}(\nu,\nu_0) \quad , \tag{8}$$

where the Lorentzian $\mathcal{L}$ is written as

$$\mathcal{L}(\nu,\nu_0) = \frac{1}{1+4(\nu_0-\nu)^2)/\Gamma^2} \quad , \tag{9}$$

in agreement with Eq. 3 of the write-up. Note that for a lightly damped oscillator $\mathcal{L}(\nu,\nu_0)$ is a sharply peaked function symmetric in frequency with respect to the maximum at resonance $\nu = \nu_0$ and having a full width half maximum (FWHM) equal to $2|\nu_0 - \nu| = \Gamma$.