

Saturated Absorption Addendum

Laser Absorption Versus Intensity Robert DeSerio

This addendum presents a simple model of how the non-saturated absorption depends on the intensity of the laser for a two-level system.

We start with the basic equations for the absorption through a thin layer dz of stationary absorbers (see Eqs. 7,8 of student writeup).

$$\frac{dI}{dz} = -\kappa I \quad (1)$$

The absorption coefficient κ is given by

$$\kappa = h\nu n\alpha(P_0 - P_1) \quad (2)$$

where n is the overall atom density and nP_0 and nP_1 are the density of atoms in the ground and upper state, respectively, and $P_0 + P_1 = 1$. The atomic absorption strength α has a Lorentzian dependence on the laser frequency ν relative to the natural resonance frequency ν_0 with a natural linewidth Γ .

$$\alpha = \alpha_0 L(\nu, \nu_0) \quad (3)$$

where

$$L(\nu, \nu_0) = \frac{1}{1 + 4(\nu - \nu_0)^2/\Gamma^2} \quad (4)$$

The population difference, $P_0 - P_1$, is given by Eq. 19 of the student writeup:

$$P_0 - P_1 = \frac{1}{1 + 2\alpha I/\gamma} \quad (5)$$

where γ is the spontaneous decay rate and is related to the Lorentzian width parameter Γ by $\gamma = 2\pi\Gamma$.

Thus we can combine the Lorentzians in α and $P_0 - P_1$ as follows:

$$\alpha(P_0 - P_1) = \frac{1}{\alpha^{-1} + 2I/\gamma} \quad (6)$$

or

$$\alpha(P_0 - P_1) = \frac{\alpha_0}{L^{-1}(\nu - \nu_0) + I/I_s} \quad (7)$$

where

$$I_s = \frac{\gamma}{2\alpha_0} \quad (8)$$

so that Eq. 2 now becomes

$$\kappa = \frac{h\nu n\alpha_0}{1 + I/I_s + 4(\nu - \nu_0)^2/\Gamma^2} \quad (9)$$

which has a Lorentzian shape with a power broadened width

$$\Gamma' = \Gamma\sqrt{1 + I/I_s} \quad (10)$$

Now we take into account the velocity distributions of atoms in the absorber gas. The density of atoms dn in a velocity group dv (where v is the component of velocity in the z -direction) is given by the Maxwell-Boltzmann distribution:

$$dn = \frac{n}{\sqrt{2\pi}\sigma_v} e^{-v^2/d\sigma_v^2} dv \quad (11)$$

with the velocity width parameter σ_v given by

$$\sigma_v = \sqrt{\frac{kT}{m}} \quad (12)$$

where T is the absolute temperature, m the atomic mass, and k is Boltzmann's constant.

The absorption per unit length $d\kappa$ for this velocity group is then obtained from Eq. 9 by replacing n with dn (from Eq. 11) and replacing the resonance frequency ν_0 in Eq. 9 with the Doppler shifter resonance frequency

$$\nu'_0 = \nu_0(1 + v/c) \quad (13)$$

This gives:

$$d\kappa = \frac{\frac{h\nu n\alpha_0}{\sqrt{2\pi}\sigma_v} e^{-v^2/2\sigma_v^2} dv}{1 + I/I_s + 4(\nu - \nu_0 - \nu_0 v/c)^2/\Gamma^2} \quad (14)$$

We now need the integral of $d\kappa$ over all velocity groups. Let

$$\Delta = \frac{\nu - \nu_0}{\Gamma/2} \quad (15)$$

and let

$$x = \frac{\nu_0 v/c}{\Gamma/2} \quad (16)$$

so

$$dv = \frac{\Gamma c}{2\nu_0} dx \quad (17)$$

and

$$d\kappa = \frac{h\nu n\alpha_0 \Gamma c}{\sqrt{2\pi}\sigma_v 2\nu_0} e^{-v^2(x)/2\sigma_v^2} dx \quad (18)$$

$$1 + I/I_s + (\Delta - x)^2$$

Now we assume $e^{-v^2(x)/2\sigma_v^2}$ is slowly varying compared to the denominator and can be pulled from the integral and evaluated at $x = \Delta$. Noting then that

$$\int_{-\infty}^{\infty} \frac{dx}{a + x^2} = \frac{\pi}{\sqrt{a}} \quad (19)$$

then gives

$$\begin{aligned} \kappa(\nu) &= \int_{-\infty}^{\infty} d\kappa \\ &= \frac{h\nu n\alpha_0 \Gamma c}{\sqrt{2\pi}\sigma_v 2\nu_0} e^{-v_\nu^2/2\sigma_v^2} \frac{\pi}{\sqrt{1 + I/I_s}} \end{aligned} \quad (20)$$

where v_ν (obtained from $x = \Delta$) is the velocity that is resonant at the laser frequency ν .

$$v_\nu = c \left(\frac{\nu - \nu_0}{\nu_0} \right) \quad (21)$$

Simplifying with $\Gamma = \gamma/2\pi$, $\alpha_0 = \gamma/2I_s$ and $i = I/I_s$ and using this κ in Eq. 2 gives:

$$\frac{di}{dz} = -\beta \frac{i}{\sqrt{1+i}} \quad (22)$$

where

$$\beta = \frac{h\nu n\gamma^2 c}{\sqrt{2\pi}\sigma_v 8\nu_0 I_s} e^{-v_\nu^2/2\sigma_v^2} \quad (23)$$

For the weak laser case $i \ll 1$, Eq. 22 reduces to

$$\frac{di}{dz} = -\beta i \quad (24)$$

which has the Beer's law behavior

$$I(z) = I_0 e^{-\beta z} \quad (25)$$

where $I_0 = I(0)$ is the incident laser intensity at the front of the cell. The absorption fraction after passage through a length L of absorber (where $I_L = I(L)$) is given by

$$\frac{I_L - I_0}{I_0} = 1 - e^{-\beta L} \quad (26)$$

which is a constant independent of the incident laser intensity.

For the strong laser case $i \gg 1$, Eq. 22 reduces to

$$\frac{di}{dz} = -\beta\sqrt{i} \quad (27)$$

which has the solution

$$I(z) = I_s \left(\sqrt{\frac{I_0}{I_s}} - \frac{\beta z}{2} \right)^2 \quad (28)$$

and the absorption fraction after passage through a length L of absorber becomes

$$\frac{I_L - I_0}{I_0} = \beta L \sqrt{\frac{I_s}{I_0}} - \frac{\beta^2 L^2 I_s}{4 I_0} \quad (29)$$

which for $I_0 \gg I_s$ decreases as $1/\sqrt{I_0}$.

The general solution to Eq. 22 should show the two limiting behaviors. Bringing the i -dependent factors to the left and dz to the right, Eq. 22 can be integrated yielding:

$$2\sqrt{1+i} + \ln \left[\frac{\sqrt{1+i} - 1}{\sqrt{1+i} + 1} \right] = -\beta z \quad (30)$$

which can be checked by implicit differentiation. Plugging in the limits of integration at the beginning and end of the cell then gives

$$\left\{ 2\sqrt{1+i} + \ln \left[\frac{\sqrt{1+i} - 1}{\sqrt{1+i} + 1} \right] \right\} \Big|_{i=i_0}^{i=i_L} = -\beta z \Big|_{z=0}^{z=L} \quad (31)$$

which becomes

$$2\sqrt{1+i_L} - 2\sqrt{1+i_0} + \ln \left[\frac{\sqrt{1+i_L} - 1}{\sqrt{1+i_L} + 1} \frac{\sqrt{1+i_0} + 1}{\sqrt{1+i_0} - 1} \right] + \beta L = 0 \quad (32)$$

Equation 32 can be solved iteratively by the Newton-Raphson method. Defining the left side of this equation as $f(i_L)$ the solution to $f(i_L) = 0$ can be found from

$$\begin{aligned} i_L(n+1) &= i_L(n) - \frac{f(i_L(n))}{\left. \frac{df}{di} \right|_{i_L(n)}} \\ &= i_L(n) - f(i_L(n)) \cdot \frac{i_L(n)}{\sqrt{1+i_L(n)}} \end{aligned} \quad (33)$$

Starting with the weak field solution $i_L = e^{-\beta L}$, the solution rapidly converges.

The solid lines in Fig. 1 are log-log plots of the absorption fraction versus the incident laser intensity obtained from the solutions (for weak laser absorption fractions of 0.95, 0.42, and 0.05) and demonstrates the proper limiting behaviors—becoming constant for low incident intensities and decreasing as $1/\sqrt{I_0}$ for high incident intensities. The figure also shows

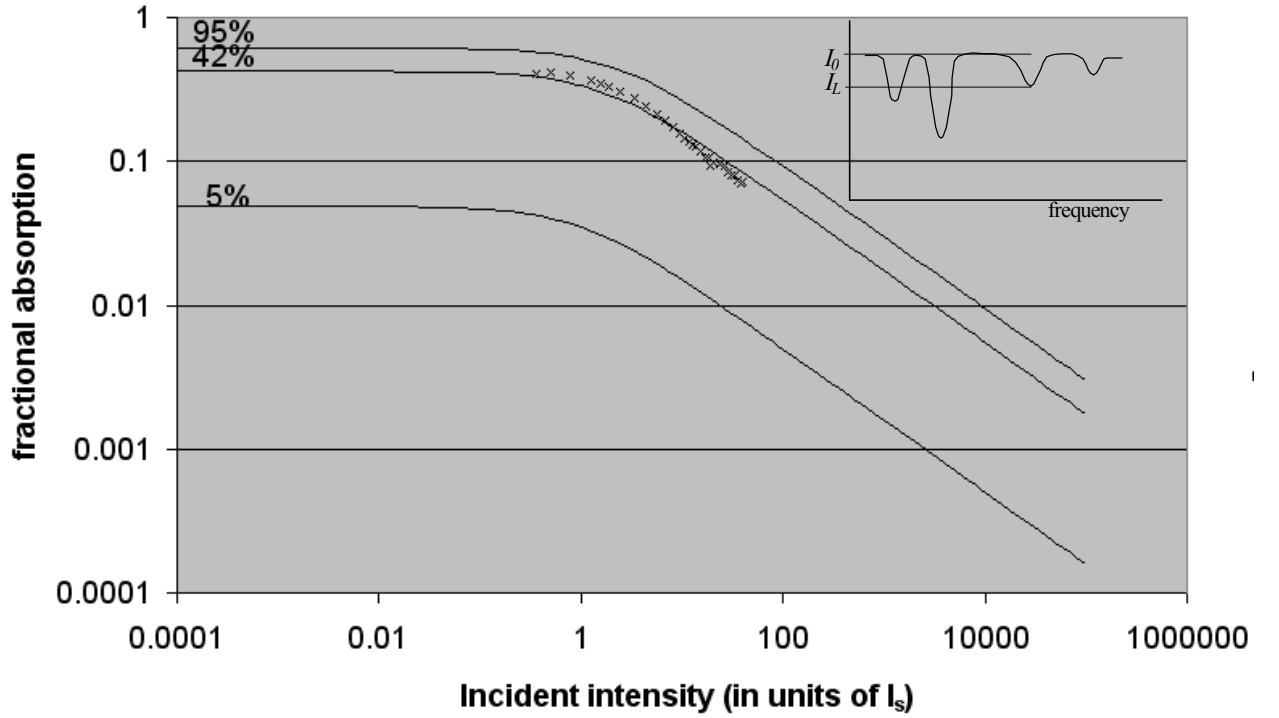


Figure 1: Log-log graphs of the predicted absorption $(I_0 - I_L)/I_0$ versus the incident intensity I_0 (in units of I_s). The solid lines show solutions for different values of β such that the low incident intensity absorption is 0.95, 0.42, and 0.05. The crosses are data from the ^{85}Rb $F = 3$ hyperfine group. The inset at the upper right shows how values proportional to I_0 and I_L were obtained.

some measurements (\times 's) for the ^{85}Rb $F = 3$ hyperfine group for our apparatus taken by Amruta Deshpande and Grace Greenlee, two PHY4803L students with I_0 and I_L taken from the absorption spectrum as illustrated in the inset. The incident intensity I_0 for the experimental data was scaled to fit the theory, with the factor so obtained in reasonable (order of magnitude) agreement with $I_s \approx 2 \text{ mW/cm}^2$ and laser intensities estimated from the detector signal, detector area and efficiency, and the laser beam diameter.