Poisson Variables

Robert DeSerio

Poisson processes

A *Poisson process* is one in which events are randomly distributed in time, space or some other variable with the number of events in any non-overlapping intervals statistically independent. For example, naturally occurring gamma rays detected in a scintillation detector are randomly distributed in time, or chocolate chips in a cookie dough are randomly distributed in volume. Here, events will be considered as randomly distributed in time so that Poisson processes will be described by a rate constant Γ.

A *homogeneous* Poisson process is one in which the event rate is constant. The expected number of events in any finite interval Δt is then given by

$$\mu = \Gamma \Delta t \tag{1}$$

A *nonhomogeneous* Poisson process is one in which the rate varies and is then considered a function of time $\Gamma(t)$. The expected number of events in an interval from t_1 to t_2 would then be the integral

$$\mu = \int_{t_1}^{t_2} \Gamma(t) dt \tag{2}$$

Due to the randomness of Poisson processes, the actual number of events n observed in the interval varies randomly with the probability for a given n depending only on μ and given by

$$P(n) = e^{-\mu} \frac{\mu^n}{n!} \tag{3}$$

This is the Poisson probability distribution; $P(n)$ gives the probability there were n events.

Two Poisson distributions for $\mu = 1.5$ and $\mu = 100$ are shown in Fig. 1. This figure shows the parent distributions. Real sample distributions would be expected to vary somewhat from the parent, getting closer to the parent as the sample size N increases.

The exponential probability density function

A better way of describing Γ is as a probability per unit time that an event will occur. That is

$$dP = \Gamma dt \tag{4}$$

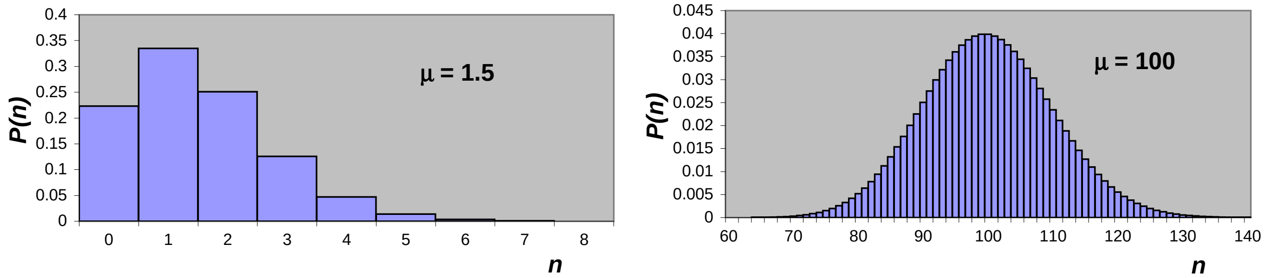

Figure 1: Poisson probabilities for means of 1.5 and 100

where dP is the differential probability that an event will occur in the infinitesimal time interval dt. Of course, some care must be taken when translating a rate to a probability per unit time. For example, if $\Gamma = 10/\mathrm{s}$, it is obviously not true that the probability is 10 that an event will occur in any particular second. However, if that same rate is expressed $\Gamma = 0.01/\mathrm{ms}$ it is roughly true that the probability is 0.01 that an event will happen in any particular millisecond. Eq. 4 only becomes exact in the limit of infinitesimal dt.

Equation 4 also describes the decay process of an excited state of an atom, nuclei, or subatomic particle. In these cases, $dP = \Gamma dt$ is the probability for the excited state to decay in the *next* time interval dt and Γ is called the decay rate for the excited state rather than an event rate.

Equation 4 leads directly to the Poisson probability distribution. The first step is to see how it leads to the exponential probability density function (pdf) giving the probability $dP_e(t)$ that the *next* Poisson event (or the decay of an excited state) will occur in the interval from t to $t + dt$.[1]

If the probability of no event (or survival of the excited state) to a time t is denoted $P(0;t)$, then the probability of no event (or survival) to $t + dt$ would be the product of this probability with the probability of no event (or no decay) in the interval dt following t. Since the probability of an event (or decay) in this interval is Γdt, the probability of no event (or no decay) in this interval is $1 - \Gamma dt$ and thus:

$$P(0;t+dt) = P(0;t)(1 - \Gamma dt) \tag{5}$$

Rearranging and substituting $(P(0;t+dt) - P(0;t))/dt = dP(0;t)/dt$ gives

$$\frac{dP(0;t)}{dt} = -\Gamma P(0;t) \tag{6}$$

which has the general solution $P(0;t) = Ae^{-\Gamma t}$. Because we must start with no event (or no decay) at $t = 0$, $P(0;0) = 1$ and so $A = 1$ giving

$$P(0;t) = e^{-\Gamma t} \tag{7}$$

[1] Equation 4 is equivalent to $dP(0) = \Gamma dt$ and must be the $t = 0$ limiting case for the general solution.

Then, the differential probability $dP(t)$ for the next event (or decay) to occur in the interval from t to $t + dt$ is given by the probability of no event (or no decay) in the interval from 0 to t followed by an event (or a decay) in the next interval dt. The former has a probability $P(0; t) = e^{-\Gamma t}$ and the later has a probability Γdt. Thus

$$dP(t) = \Gamma e^{-\Gamma t} dt \tag{8}$$

Equations 8 is a continuous probability density. It is properly normalized, i.e., the integral over all times from 0 to ∞ is unity as required. It also has the very reasonable property that the expectation value for the random variable t—the time to the next event (or to the decay)—is given by

$$\begin{aligned} \langle t \rangle &= \int_0^\infty t \Gamma e^{-\Gamma t} dt \\ &= \frac{1}{\Gamma} \end{aligned} \tag{9}$$

In the case of decay, the *lifetime,* τ, of the excited state is defined as the expectation value $\langle t \rangle$ and is another way to quantify the Poisson process. If the decay rate is 1000/s, the lifetime is 0.001 s. Equation 8 is sometimes expressed in terms of the lifetime rather than the decay rate.

$$dP(t) = \frac{1}{\tau} e^{-t/\tau} dt \tag{10}$$

The probability for decay in a time τ is found by integrating Eq. 8 from 0 to τ and gives the value $1/e$. Thus, for a large sample of excited states at $t = 0$, the fraction $1/e$ of them will have decayed by τ. The time it would take for half the sample to decay is called the half-life and is easily shown to be $\tau \ln 2$.

The Poisson probability distribution

There are several possible derivations of the Poisson probability distribution. It is often derived as a limiting case of the binomial probability distribution. The derivation to follow relies on Eq. 4 and begins with Eq. 7 for the probability $P(0; t)$ that there will be no events in some finite interval t.

Next, a recursion relation is derived for the probability, denoted $P(n + 1; t)$, for there to be $n + 1$ events in a time t, which will be based on the probability $P(n; t)$ of one less event. For there to be $n + 1$ events in t, three independent events must happen in the following order (their probabilities given in parentheses).

- There must be n events up to some point t' in the interval from 0 to t ($P(n; t')$ by definition).
- An event must occur in the infinitesimal interval from t' to $t' + dt'$ ($\Gamma dt'$ by Eq. 4).
- There must be no events in the interval from t' to t ($P(0, t - t')$ by definition).

The probability of $n+1$ events in the interval from 0 to t would be the product of the three probabilities above integrated over all t' from 0 to t to take into account that the last event may occur at any time in the interval. That is,

$$P(n+1;t) = \int_0^t P(n;t')\Gamma dt' P(0;t-t') \tag{11}$$

From Eq. 7 we already have $P(0;t-t') = e^{-\Gamma(t-t')}$ and thus Eq. 11 becomes

$$P(n+1;t) = \Gamma \int_0^t P(n;t') e^{-\Gamma(t-t')} dt' \tag{12}$$

With the following definition:

$$P(n;t) = e^{-\Gamma t}\overline{P}(n;t) \tag{13}$$

Eq. 12 becomes

$$\begin{aligned} e^{-\Gamma t}\overline{P}(n+1;t) &= \Gamma \int_0^t e^{-\Gamma t'}\overline{P}(n;t') e^{-\Gamma(t-t')} dt' \\ \overline{P}(n+1;t) &= \Gamma \int_0^t \overline{P}(n;t') dt' \end{aligned} \tag{14}$$

Differentiating with respect to t then gives

$$\frac{d\overline{P}(n+1;t)}{dt} = \Gamma \overline{P}(n;t) \tag{15}$$

With the solution

$$\overline{P}(n;t) = \frac{(\Gamma t)^n}{n!} \tag{16}$$

And thus with Eq. 13, the Poisson probabilities become

$$P(n;t) = e^{-\Gamma t}\frac{(\Gamma t)^n}{n!} \tag{17}$$

Note that, as expected, the right side depends only on the combination $\mu = \Gamma t$. Using that substitution and dropping the now meaningless t dependence on the left, the standard Poisson probability distribution of Eq. 3 results.

Although the Poisson probabilities were derived assuming a homogeneous process, they are also appropriate for nonhomogeneous processes with the appropriate value of μ (Eq. 2).