PH6246, Section 3916, Fall 2015, Homework 4

Due at the start of class on Friday, October 2.

Answer all questions. Please write neatly and include your name on the front page of your answers. To gain maximum credit you should explain your reasoning and show all working.

1. A particle of mass m is constrained to move under gravity without friction the inside of a paraboloid of revolution whose axis is vertical.

   a) Write down the Lagrangian, modified to include the equation of constraint, and find the equations of motion for all three coordinates in the constrained system.

   b) Find the one dimensional problem equivalent to its motion.

   c) What is the condition on the particles initial velocity to produce circular motion?

   d) Find the period of small oscillations about this circular motion.

2. The most general rotation matrix can be written in terms of Euler angles as:

$$\mathbf{A} \equiv a^i_{\ j} = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}.$$

   a) Show that this follows from $\mathbf{A} = \mathbf{BCD}$, where $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are:

$$\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   respectively.

   b) Show that $a^i_{\ j} a^k_{\ l} g_{ik} = g_{jl}$, where, in cartesian coordinates,

$$\mathbf{g} \equiv g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Note: you cannot use simple matrix multiplication because g_{ij} is not a matrix.

   c) Now consider the matrix $\mathbf{F} \equiv f^{\ j}_{i}$ given by the transpose of matrix $\mathbf{A}$. Show that $\mathbf{FA} = \mathbf{AF} = \mathbf{I}$ (where $\mathbf{I}$ is the unit matrix), or:

$$f^i_{\ j} a^j_{\ k} = a^i_{\ l} f^l_{\ k} = \delta^i_{\ k}.$$

   Note: this result follows by simple matrix multiplication, in contrast to part b).

   d) How are there two seemingly different results related to the orthogonality of $\mathbf{A}$?

3. Consider the equation $\mathbf{G} = \mathbf{AF}$ under a transformation $\mathbf{B}$: *i.e.,* $\mathbf{G}' = \mathbf{BG}$ and $\mathbf{F}' = \mathbf{BF}$.

a) Show that, in order for $\mathbf{G}' = \mathbf{A}'\mathbf{F}'$ to hold, we must have $\mathbf{A}' = \mathbf{BAB^{-1}}$.

b) Show that $\det \mathbf{A}' = \det \mathbf{A}$.

c) Show that $\operatorname{trace} \mathbf{A}' = \operatorname{trace} \mathbf{A}$.

d) Let $\mathbf{B}$ be the transformation which takes $\mathbf{A}$, an arbitrary rotation matrix, to the specific one $\mathbf{A}'$ defined by the condition that it replaces $\mathbf{A}$ by a single rotation about a vector fixed in the body: *i.e.,*

$$\mathbf{A}' = \begin{bmatrix} \cos\Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Show that the eigenvalues of $\mathbf{A}'$ are $\{1, \exp i\Phi, \exp -i\Phi\}$.

e) By considering the trace $\mathbf{A}'$, and for $\mathbf{A}$ given as in problem 2, show that we have:

$$\cos\frac{\Phi}{2} = \cos\frac{\theta}{2}\cos\frac{\phi+\psi}{2}.$$

4. In a non-inertial frame on the surface of the Earth, the Centrifugal Force is perpendicular to the axis of rotation. Let g_0 be the purely gravitational force at the surface of the Earth and assume that the Earth is exactly spherical.

a) Show that, at the Poles, $g_P = g_0$ and points toward the Earth's center.

b) Show that, at the Equator, $g_E = g_0 - \Omega^2 R_e$ and points toward the Earth's center.

c) Calculate the percentage difference, assuming $g_0 \equiv 9.8$ m/s^2 and $R_e \approx 6,400$ km.

d) At an arbitrary point on the Earth's surface, the net force on a stationary mass does not point directly toward the Earth's center. Using an appropriate cartesian frame at the surface of the Earth, find the various components of the net force on a stationary mass due to gravitational and centrifugal effects.

e) Show that the magnitude of the net force at an angle θ from the Pole can be written as $g_\theta = \sqrt{g_P^2\cos^2\theta + g_E^2\sin^2\theta}$.

5. The Coriolis Force only acts on masses when they are in motion. Consider a mass of air which has initially been set in motion by a pressure difference.

a) Suppose the air-mass is no longer subject to a horizontal pressure force, and that it remains at a fixed distance above the surface of the earth. Write down the equations governing its motion.

b) Solve these (coupled) equations for generic initial data.

c) If the air-mass is originally moving East at latitude 45° N, show that in less than 4.25 hours the airmass will be moving in a southerly direction.

d) Is the effect of the Coriolis force greater at the Equator or at the Poles? Give an explanation for your answer.