PH6246, Section 3916, Fall 2015, Homework 9

Due at the start of class on Friday, November 13.

Answer all questions. Please write neatly and include your name on the front page of your answers. To gain maximum credit you should explain your reasoning and show all working.

1. In an inertial frame at rest, a rocket with mass m is seen to have speed v. At $t = 0$, the rocket begins to thrust by emitting exhaust backwards relative to the rocket. In a short interval of time dt, the rocket changes its mass from m to $m + dm$ (where $dm < 0$) and its speed from v to $v + dv$ (where $dv > 0$), while it emits exhaust at speed v_{ex} backwards relative to the rest frame of the rocket.

a) Write down the equation for the conservation of total four-momentum of the system. How many independent equations do you have? What are the unknowns?

b) Using the conservation of relativistic four-momentum, find an equation for dv in terms of m, dm, v, v_{ex} and c.

c) Hence, write down a differential equation for v as a function of m, and solve it to find $v(m)$.

d) Find the mass of the exhaust material which is emitted in the time interval dt.

2. Consider a relativistic particle of mass m moving in one spatial dimension, and suppose that the Lagrangian is given by:

$$L = -mc\sqrt{c^2 - \dot{x}^2} + max.$$

a) Write down the Euler-Lagrange equation governing the motion of the particle.

b) Solve the equation of motion for arbitrary initial data and show that the orbit of the particle is hyperbolic in the (x, t) plane.

c) Find the equations for the two asymptotes of the hyperbola, and the coordinates of their point of intersection.

d) Suppose that the particle starts from rest at the origin. After what time t_0 would a photon released from the origin be never able to reach the particle?

3. In an inertial frame at rest (the lab frame), a photon of energy E_1 collides with another photon of energy E_2, where that angle between their initial trajectories is θ.

a) Write down an expression for the total four-momentum of the system before the collision.

b) What is the value of the squared length of this four-momentum in the lab frame and in the center of momentum frame?

c) What is the actual four-momentum of the two photon system in the center of momentum frame?

d) The photons annihilate and form two identical particles each of mass m. What do you know about the spatial components of the four-momenta of each particle in the center of momentum frame?

e) For this interaction to take place, the energy of the photons must satisfy $\sqrt{E_1 E_2} \geq E_{\text{min}}$. Find an expression for E_{min} in terms of $E_0 = mc^2$, the rest energy of each particle, and θ, that angle between the initial photons.

4. The Lagrangian for a relativistic charged particle of mass m and charge q moving in an external electromagnetic field, can be written in a reparameterization invariant form. By choosing the parameter to be coordinate time, *i.e.*, $d/d\lambda = d/dt = \dot{}$, the Lagrangian is given by:

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = -mc\sqrt{c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2} + q(-V + \dot{x}A_x + \dot{y}A_y + \dot{z}A_z),$$

where the scalar V and the components A_i of the vector $\mathbf{A}$ are all functions of (x, y, z, t).

a) Write down the canonical momentum for each of the three coordinates.

b) Solve for $\sqrt{c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}$ entirely in terms of the canonical momenta p_i, the vector potential A_i, and m and c.

c) Hence write down the Hamiltonian for the system.

d) Consider the vector $Þ^\mu \equiv mu^\mu + qA^\mu$ and find H in terms of the zeroth component of Þ.

e) For the revised Lagrangian (now with $\lambda = \tau$, so that $\dot{}$ now equals $d/d\tau$):

$$L' \equiv L\dot{t} = -mc\sqrt{c^2\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2} + q(\dot{t}A_t + \dot{x}A_x + \dot{y}A_y + \dot{z}A_z),$$

show that the Hamiltonian is identically zero.

5. Consider a Lagrangian of the form:

$$L = \frac{1}{2}\left(m\dot{x}^2 - kx^2\right)e^{\gamma t},$$

where the particle of mass m moves in one dimension. Assume k is positive.

a) Find the equation of motion for the particle, and solve it for a particle which is at x_0 with velocity v_0 at $t = 0$.

b) Interpret the equation in terms of forces acting on the particle.

c) Find the canonical momentum and construct the Hamiltonian. Is it a constant of the motion?

d) Solve Hamilton's equations for the motion of the particle. For the initial data given in a), what happens to the canonical coordinate $x(t)$ and the canonical momentum $p(t)$ as t approaches infinity?