PHY6246, Section 3916, Fall 2016, Homework 2

Due at the start of class on Monday, September 12.

Answer all questions. Please write neatly and include your name on the front page of your answers. To gain maximum credit you should explain your reasoning and show all working.

1. The one-dimensional harmonic oscillator has the Lagrangian $L = m\dot{x}^2/2 - kx^2/2$. Suppose you did not know the solution to the motion, but realized that the motion must be periodic, and therefore could be described by a Fourier series of the form:

$$x(t) = \sum_{j=0} a_j \cos j\omega t.$$

(taking $t = 0$ as a turning point) where ω is the unknown angular frequency of the motion. This representation for $x(t)$ defines a many-parameter path for the system point in configuration space. Consider the action integral I for two points $t_)1$ and t_2 separated by the period $T = 2\pi/\omega$. Show that with this form for the system path, I is an extremum for non-vanishing x only if $a_j = 0$, for $j \neq 1$, and only if $\omega^2 = k/m$.

2. A disk of radius R rolls without slipping inside the stationary parabola $y = ax^2$.

   a) Find the equation of constraint.

   b) What condition allows the disk to roll so that it touches the parabola at one and only one point, independent of its position?

3. This problem concerns a central force given by a potential $V(r)$.

   a) Show that, if a particle describes a circular orbit under the influence of an attractive central force directed toward a point on the circle, then the force varies as the inverse-fifth power of the distance.

   b) Show that for the orbit described, the total energy of the particle is zero.

   c) Find the period of the motion.

   d) Find $\dot{x}$, $\dot{y}$ and v as a function of the angle around the circle and show that all three quantities are infinite as the particle goes through the center of force.

4. This problem concerns conic sections.

   a) For circular and a parabolic orbits in an attractive $1/r$ potential having the same angular momentum, show that the perihelion distance of the parabola is one half the radius of the circle.

b) Prove that in the same central force as in part a) the speed of a particle at any point in a parabolic orbit is $\sqrt{2}$ times the speed of a circular orbit passing through the same point.

5. A particle moves in a force field defined by the Yukawa potential:

$$V(r) = -\frac{k}{r}\exp\left(-\frac{r}{a}\right),$$

where both k and a are positive.

a) Write the equations of motion and reduce them to the equivalent one-dimensional problem. Use the effective potential to discuss the qualitative nature of the orbits for different values of the energy and angular momentum.

b) Show that if the orbit is nearly circular, the apsides will advance approximately by $\pi\rho^2/a^2$ per revolution, where ρ is the radius of the circular orbit.