PHY6246, Section 3916, Fall 2016, Homework 12

Due at the start of class on Monday, November 21.

Answer all questions. Please write neatly and include your name on the front page of your answers. To gain maximum credit you should explain your reasoning and show all working.

1. Consider the following 'complete' solution to some Hamilton-Jacobi equation:

$$S(x;\gamma;t) = \frac{m}{2}\left\{\frac{(x-\gamma)^2}{t} + (x+\gamma)gt - \frac{g^2t^3}{12}\right\}.$$

   a) Identify the momentum p conjugate to x and find the Hamiltonian, showing fully your procedure. What mechanical system does this Hamiltonian describe.

   b) Solve the dynamical equations completely, for both x and its canonical momentum, clearly identifying all the quantities you need to define.

   c) Identify γ and $\partial S/\partial\gamma$ in terms of the initial data for x and p, and comment on your findings. How would you argue that there is sufficient information to interpret S as the value of an action for some boundary data?

2. A certain physical system is described by the following Lagrangian:

$$L = -B\dot{C} - V(C) - \frac{B^2}{2}.$$

   a) Calculate the canonical momenta for B and C and show that the system is governed by the two constraints:

$$\Psi_1 = P_B = 0, \qquad \text{and} \qquad \Psi_2 = P_C + B = 0.$$

   b) Show that transformation to the new variables:

$$Q_1 = B + P_C, \quad P_1 = P_B, \quad Q_2 = C + P_B, \quad P_2 = P_C$$

   is canonical and that, in terms of these new variables, the canonical Hamiltonian depends only on the second pair when the constraints are satisfied.

   c) For the new Hamiltonian:

$$H_T = V(c) + \frac{B^2}{2} + \frac{\partial V}{\partial C}P_B - B(P_C + B),$$

show, by the equations of motion it gives rise to, that P_B and $P_C + B$ both remain zero if they start out zero, and that the equations for C and P_C are then as would follow in part b) above.

3. The logistic equation, given by $x_{n+1} = ax_n(1 - x_n)$, is an example of a one dimensional discontinuous, nonlinear system which exhibits chaos.

   a) In the domain where the solution reaches a limit as $n \Rightarrow \infty$, find the limit.

   b) Outside a certain interval for the parameter a, the solution does not converge stably to this limit. Find the range of $a_{\text{min}} < a < a_{\text{max}}$ for which the limit is stable.

   c) Consider some value for a slightly greater than a_{max}, the larger of the two limit values you found in part b). For the parameter a in this domain, the solution as $n \Rightarrow \infty$ oscillates between two fixed values (dependent on the value of a). Show that a cubic equation needs to be solved to find these two values.

   d) Show that your solution in part a) also satisfies the cubic equation. Hence, find expressions for these two fixed values between which the final solution to the logistic equation oscillates.

   e) As the parameter a is increased still further, a limit is reached, beyond which the solution begins to oscillate between four distinct values. Find an expression for the limiting value of a beyond which this behavior occurs. Show that the limiting value is approximately given by $a = 3.4495$. (Machine algebra is allowed).

4. This problem concerns the motion of a free, point particle moving in one dimension.

   a) Solve the Hamilton-Jacobi equation for $S(q, \alpha, t)$, where α is the initial momentum of the particle.

   b) By finding the momentum p conjugate to q, and the constant β conjugate to α, solve for $q(t)$ and $p(t)$.

   c) Now suppose that the Hamiltonian is "perturbed" by $\Delta H = \frac{1}{2}kq^2$, where k is presumed to be small. Use time dependent perturbation theory to find the perturbed solution to second order (*i.e.,* iterate twice), assuming $q(0) = q_0 \neq 0$ and $p(0) = 0$.

   d) What is the effective expansion parameter in the series you obtain? Identify the two series expansions you obtain as containing the first three terms in the Taylor expansion of well known trigonometric functions.

e) Explain why you expect the infinitely iterated solution to hold even when k is not small.

5. A mass point m hangs at one end of a vertically hung Hooke's law spring of force constant k.

   a) Solve the Hamilton-Jacobi equation for $S(q, \alpha, t)$, where α corresponds to the energy of the system.

   b) By finding the momentum p conjugate to q, and the constant β conjugate to α, solve for $q(t)$ and $p(t)$.

   c) Now suppose that the suspension-point of the spring oscillates vertically according to $z = a\cos(\omega_1 t)$. By treating a as a small quantity, obtain a first-order (in a) solution for the motion of m in time, using time dependent perturbation theory.

   d) What happens as ω_1 approaches the unperturbed frequency ω_0?