PHY6246, Section 3916, Fall 2017, Homework 13

Due at the start of class on Monday, December 4.

Answer all questions. Please write neatly and include your name on the front page of your answers. To gain maximum credit you should explain your reasoning and show all working.

1. Consider the problem of a point projectile moving in a vertical plane under the influence of a uniform gravitational field. Assume the projectile starts at the origin at $t = 0$, with velocity v, and that it is launched at an angle α to the horizontal.

   a) Write down a Lagrangian describing the motion, and find the Hamiltonian.

   b) Write down and solve the Hamilton-Jacobi equation.

   c) Hence find, fully, the equation for the trajectory.

   d) Similarly, find the dependence of the coordinates on time during the flight of the projectile.

2. Consider the problem of a point particle moving in a quadratic potential subject to a small quartic perturbation.

   a) Write down a Lagrangian describing the motion, and find the Hamiltonian.

   b) For the unperturbed Hamiltonian, express the canonical coordinates (q and p) and the Hamiltonian in terms of action-angle variables.

   c) For the perturbed Hamiltonian, write down the perturbation in action-angle variables..

   d) Write down and solve the equations of motion, to first order in the perturbation. (Hint: use time-dependent perturbation theory).

   e) Comment on the effects of the perturbation upon the periodicity of the motion for this system.

3. The Lagrangian density for a charged scalar meson field can be written as:

$$\mathcal{L} = \dot{\phi}\dot{\phi}^* - c^2 \nabla\phi \nabla\phi^* - \mu_0^2 c^2 \phi\phi^*,$$

in which ϕ and ϕ^* are to be taken as two independent field variables.

   a) Write out the Euler-Lagrange equations for this system, and obtain the equation of motion for ϕ.

b) Find the canonical momenta, and obtain the complete Hamiltonian density.

c) What physical dimension does μ_0 have? What physical (quantum ?) characteristic of the meson field might it represent?

d) Write out an expression for the conserved current j_μ, and explain why it might be a valid current to include as a source term in Maxwell's Equations.

e) What property of the original Lagrangian density allows this current to be a conserved quantity?

4. A crucial step in showing that the waves in a fluid are necessarily longitudinal requires establishing that:

$$\int f'(\mathbf{n} \cdot \mathbf{r} - ct)dt = -\frac{1}{c} f(\mathbf{n} \cdot \mathbf{r} - ct).$$

a) For an arbitrary function $f(\xi)$, with derivative $f'(\xi)$, prove this.

b) Show that if we make the change of variables $\xi = x - ct$ and $\eta = x + ct$ then:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = -4c^2 \frac{\partial}{\partial \xi} \frac{\partial u(x(\xi,\eta), t(\xi,\eta))}{\partial \eta},$$

and hence write down a complete solution for a wave on an infinite string.

c) Consider a string of length L and clamped at both ends. At $t = 0$, suppose that the displacement of the string is given by $u(x,0) = u_0 \sin(n\pi x/L)$ where n is an integer and its instantaneous velocity satisfies $\dot{u}(x,0) = 0$. By the separation of variables (or some other approach) find the full solution describing the motion of the string.

5. Consider the Laplacian:

$$\nabla^2 f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^j} \right).$$

a) Show that, in spherical polar coordinates (with $g_{ij} = \mathrm{diag}(1, r^2, r^2 \sin^2\theta)$), this is equivalent to:

$$\nabla^2 f = \frac{1}{r} \frac{\partial^2 r f}{\partial r^2} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \phi^2}.$$

b) Since waves in three dimensions obey:

$$\frac{\partial^2 f(x^i,t)}{\partial t^2} - c^2 \nabla^2 f(x^i,t) = 0,$$

show that a spherical light wave (away from the origin $r = 0$) can be given by:

$$f(x^i, t) = \Re\left(\frac{F_0 \exp i(kr \pm \omega t)}{r}\right),$$

where f is any component of $\mathbf{A}$, and find the relation between k and ω.

c) There can be an infinite number of such solutions. What quantities can be varied in creating sums of such solutions? Argue that, if the integral exists and is sufficiently differentiable, then:

$$f(x^i, t) = \Re \int_{-\infty}^{\infty} \frac{F(\omega) \exp i(k(\omega) r - \omega t)}{r} d\omega,$$

is also a solution (away from $r = 0$).