CONSTRAINTS: notes by BERNARD F. WHITING

Whether for practical reasons or of necessity, we often find ourselves considering dynamical systems which are subject to physical constraints. In such situations it is possible to consider redefining the dynamical variables in a theory so that constrained and unconstrained degrees of freedom become decoupled. For situations where that separation cannot be carried out explicitly, or it is undesirable to do so, Dirac has introduced a construction which acts as an effective Poisson bracket on the physical phase space, in which the constrained degrees of freedom can be essentially eliminated. In these notes we shall look basically at methods for categorizing and dealing with constraints in dynamical systems, noticing an important distinction between constraints of different type, and illustrate Dirac's procedure with a simple example. In fact, as we shall see, most points which need to be made can be demonstrated very well with simple examples.

A complete understanding of constrained dynamical systems requires a thorough knowledge of classical Hamiltonian theory. By the use of a number of carefully selected examples I will attempt to show how this knowledge is used, and how it may be usefully extended without deviating from the spirit of the traditional Hamiltonian approach. In order to illustrate the difference in the kinds of constraints that arise, we can begin with two examples of problems which should be quite familiar. Then a minimal set of skills necessary to proceed will be briefly described. Finally simple examples showing how these procedures can be put into practice will be given.

Examples distinguishing constraint types

i) Constraints form canonical pairs

This example concerns particle motion confined to the surface of a sphere. We suppose that the Lagrangian for the system is given by:

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z),$$

subject to the constraint

$$\phi_1 = \sqrt{x^2 + y^2 + z^2} - a \approx 0 \ ,$$

(where ≈ 0 implies "is zero when the constraint (ϕ_1) is imposed") which fixes the radius of the sphere. The conjugate momenta for this system are defined by:

$$p_x = m\dot{x} \ , \quad p_y = m\dot{y} \ , \quad p_z = m\dot{z} \ ,$$

which give rise to the canonical Hamiltonian

$$\mathcal{H}_c = p\dot{q} - \mathcal{L} = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + V(x, y, z) \ .$$

However, for this constrained system we take the canonical generator of time translations to be the primary Hamiltonian given by:

$$\mathcal{H}_p = \mathcal{H}_c + \lambda\phi_1 \ ,$$

where λ is taken as a Lagrange multiplier [Note: λ is not considered to be a dynamical variable and it has no canonically conjugate momentum, but variation of $\mathcal{H}_p$ with respect to λ enforces the constraint $\phi_1 = 0$.]

Before proceeding, we need to determine whether there are any additional constraints arising from the requirement that $\phi_1 = 0$ be maintained in time. To this end, we compute:

$$\dot{\phi}_1 = \{\phi_1 \ , \mathcal{H}\} = \frac{1}{m\sqrt{x^2+y^2+z^2}}\left(xp_x + yp_y + zp_z\right) \ ,$$

where by definition,

$$\{x(q,p), y(q,p)\} \equiv \frac{\partial x}{\partial q^i}\frac{\partial y}{\partial p_i} - \frac{\partial x}{\partial p_i}\frac{\partial y}{\partial q^i} \ .$$

Since $\dot{\phi}_1$ does not automatically vanish, we must impose it as an additional constraint:

$$\phi_2 = \frac{1}{\sqrt{x^2+y^2+z^2}}\left(xp_x + yp_y + zp_z\right) \approx 0 \ .$$

Then

$$\begin{aligned}\dot{\phi}_2 &= \{\phi_2 \ , \mathcal{H}_p\} \\ &= \{\phi_2 \ , \mathcal{H}_c\} + \lambda\{\phi_2 \ , \phi_1\} \ ,\end{aligned}$$

in which we can always impose $\dot{\phi}_2 = 0$ by solving for λ (which, in general, will be time dependent), since $\{\phi_1 \ , \phi_2\} = 1$ is non-zero. In fact we see that even though ϕ_1 and ϕ_2 should be zero, they satisfy the conditions of a pair of canonical coordinates on phase space: but in this formulation ϕ_1 and ϕ_2 are not independent of the other canonical variables x, y, z and p_x, p_y and p_z. Can we effect a canonical transformation which disentangles constraint degrees of freedom from dynamically non-trivial degrees of freedom?

We could have eliminated all necessity of constraints in this problem by starting in spherical polar coordinates with $r = a$, and $\dot{r} = 0$ imposed explicitly in the Lagrangian, so the answer to our question should be an obvious 'yes.' However, to get a feel for what the question involves it's useful to reformulate this particular problem in spherical polar coordinates while maintaining the original constraint. Thus, for the Lagrangian we would have:

$$\mathcal{L} = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - V(r,\theta,\phi) \ .$$

and for the constraint:

$$\phi_1 = r - a = 0 \ .$$

The canonical momenta are now given by

$$p_r = m\dot{r} \ , \quad p_\theta = mr^2\dot{\theta} \quad \text{and} \quad p_\phi = mr^2 \sin^2\theta\dot{\phi} \ ,$$

so that $\mathcal{H}_c$ becomes

$$\frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + V(r,\theta,\phi) \ ,$$

while the definition of $\mathcal{H}_p$ in terms of $\mathcal{H}_c$ and ϕ_1 remains unchanged. Similarly

$$\dot{\phi}_1 = \{\phi_1 \ , \mathcal{H}_p\} = p_r \ ,$$

so that ϕ_2 simplifies considerably to

$$\phi_2 = p_r \ .$$

Then

$$\dot{\phi}_2 = \{\phi_2, \mathcal{H}_c\} + \lambda$$

which can again be solved for λ. Now, however, we see that our canonical variables can be split into two groups: θ, p_θ, ϕ and p_ϕ which are now unconstrained; and $\phi_1 = r - a$, $\phi_2 = p_r$, which have decoupled from the remainder, and are both constrained to vanish. In other words, we have found the answer to the question posed above. We now consider another familiar example.

ii) Conjugate of constraints totally unconstrained

The most important observation here is that the constraints will be of a different type. This second example is electromagnetism, for which we consider the Lagrangian density to be (cartesian coordinates in flat space will be assumed throughout):

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \lambda j^\mu A_\mu$$

in which $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and λ is now a coupling constant, NOT a Lagrange multiplier. In this Lagrangian density, A_0 has no time derivative, so the definition $\Pi^\mu = \frac{\partial\mathcal{L}}{\partial\dot{A}_\mu}$ will introduce the constraint

$$\phi^0 = \Pi^0 \approx 0 \ .$$

The canonical Hamiltonian density can be shown to be

$$\mathcal{H}_c = \frac{1}{2}\Pi_i\Pi^i + \Pi^i\partial_i A_0 + \frac{1}{4}F_{ij}F^{ij} - \lambda j^\mu A_\mu \ ,$$

while the Hamiltonian density generating canonical time translations is

$$\mathcal{H}_p = \mathcal{H}_c + \vartheta(x^\mu)\phi^0 \ .$$

Then

$$\dot{\phi}^0 = \{\phi^0, \mathcal{H}_p\} = \partial_i \Pi^i + \lambda j^0 \ ,$$

which requires us to introduce the additional constraint

$$\phi^1 = \partial_i \Pi^i + \lambda j^0 \approx 0 \ .$$

For the time derivative, we now compute:

$$\dot{\phi}^1 = \{\partial_i \Pi^i, \mathcal{H}_p\} + \lambda \partial_0 j^0 = \lambda \partial_\mu j^\mu$$

In this case we cannot solve for the Lagrange multiplier $\vartheta(x^\mu)$, but must instead restrict our attention to sources whose current is conserved. Note in particular that

$$\{\phi^0, \phi^1\} = 0 \ ,$$

So now ϕ^0 and ϕ^1 do not form a canonical pair. In fact if we decompose A_i and Π^i into (orthogonal) transverse and longitudinal components

$$A_i = A_i^T + A_i^L \quad \text{and} \quad \Pi^i = \Pi^{Ti} + \Pi^{Ti}$$

(where V^{Ti} satisfies $\partial_i V^{Ti} = 0$), A_i^T, Π^{Ti} represent two dynamical degrees of freedom, while (A_0, Π^0) and $(A, \partial_i \Pi^{Li})$ (where $A_i^L = \partial_i A$) represents decoupled degrees of freedom in which $\Pi^0 = \phi^0$ and $\partial_i \Pi^{Li} = \phi^1 - \lambda j^0$ are given by constraints, while canonical time evolution can tell us nothing at all about the time dependence of A_0, nor A if we follow the gauge theory approach indicated below. With this decomposition <u>all</u> other relevant Poisson brackets vanish and (up to integration by parts) the Hamiltonian density can be broken into decoupled disjoint parts

$$\mathcal{H} = \mathcal{H}_T + \mathcal{H}_L \ .$$

The constraints here are of a different type to those in the previous example, because they no longer form conjugate pairs; instead, their conjugates dynamically decouple from the physical degrees of freedom and their time dependence may no longer be governed by the dynamical equations of the theory.

Handling constraints in dynamical systems

In the first example considered above, we could use our knowledge of the problem to achieve a definite split between the dynamical and non dynamical degrees of freedom. This splitting, once obtained, essentially allows us to discard the constrained canonical variables. Practitioners have seen that simple problems like this could indicate a general principle to pursue with constraints of this type. In the second example, the split was perhaps much less obvious - it is actually considerably clearer in momentum space - and the different character of the constraints is related to the deep rooted property that electromagnetism couples to conserved currents. Since it occupies such an important place in Physics, electromagnetism as a gauge theory is often taken as the archetypical model for systems with this kind of constraints. (Although electromagnetism is a field theory, the theory discussed in Example 2a below, has the same kind of constraints with finite degrees of freedom.)

In all situations, the principle of disentangling will be the same: i.e. one tries to so categorize the dynamical variables and the relations between them that a subset of them can be regarded as obeying an unconstrained dynamics, while the remainder decouple and become dynamically - if not physically - irrelevant (at least in classical mechanics). Fortunately, Dirac has put forward a formulation which deals with the situation where the split is too difficult to achieve explicitly, or where it may even be technically impossible. Before going on to consider Dirac's formulation it will be useful to establish more clearly the distinction between the two types of constraint which we have already encountered.

Although usually not described quite this way, the classification of constraints I will give makes most sense if we can imagine that the dynamical and constrained degrees of freedom have already been separated, and that each group can really be re-organized into distinct canonical pairs. Then,

when a pair of canonical variables p, q st $\{q, p\} = 1$ are both constraints, we describe the constraints as being second class. When, in a canonical pair, only one variable is a constraint, we describe that constraint as being first class. Only true dynamical degrees of freedom commute with all the constraints.

Up to this point, what has been said is consistent with the usage given by most authors in the field. Even the terminology I have used so far is not standard in the literature, but beyond this point also the general procedure to follow is not universally accepted. Some would have us introduce additional 'gauge fixing' conditions, $C^a(q,p) = 0$, to convert all first class constraints into second class constraints. In other situations we would be persuaded to introduce extra degrees of freedom so that all second class constraints can be transformed into first class constraints (this may lead to some modification of the effect of the original constraints - see the comments concerning Example 3 below), and then the whole problem can be treated as an extended gauge theory. In either case, the purpose is to allow eventually for a uniform treatment of the adjusted system.

For concreteness, I will follow a development compatible with Dirac's, though not always following his exquisite logic. As one can see, the main objective indicated is to come up with a well defined set of quantities, the (true) dynamical variables of the reduced phase space, which commute with all the constraints. When such a split is not possible in the second class case, we shall follow Dirac directly and introduce a modified bracket (it *is* actually a Poisson bracket on the reduced phase space), which serves to govern the dynamics in the same way that the Poisson bracket does for unconstrained systems. To define Dirac's bracket we must first define the 'matrix' of Poisson brackets for all the second class constraints:

$$C^{ij} = \{\phi^i, \phi^j\} \ ,$$

which is non-singular and therefore invertible. In terms of its inverse, $(C^{-1})_{ij}$, we can then define the Dirac bracket:

$$\{A, B\}_{DB} = \{A, B\} - \{A, \phi^i\}(C^{-1})_{ij}\{\phi^i, B\} \ ,$$

This does not depend, for its definition, on the constraints being isolated from the remaining dynamical variables, nor does it require that they already be organized into canonical pairs. (However, we do have to be able to distinguish the first from the second class constraints, since the former would prevent the matrix of Poisson brackets from being invertible.) Furthermore, this definition has the obvious property that the Dirac bracket of a constraint with anything else will always be zero. Thus, all physical quantities 'Dirac bracket commute' with all constraints, a condition which renders the constraints classically irrelevant, even if they cannot be separated explicitly from the true degrees of freedom.

Nest, we discuss the first class case. There is a general tendency to attempt to interpret first class constraints as generates of gauge transformations, though Example 2a below provides a counterexample. Since gauge theories do occur frequently in physics, we need to be able to handle them, and since one is often required to fix a gauge even in classical theory, I will consider this approach briefly using electromagnetism in Example 4 below. Normally, we would attempt to decouple first class constraints and true dynamical degrees of freedom whenever it is possible to do so. When that is not possible, resort to gauge fixing may become desirable, but then we often have to tackle the problem of the non-existence of a global gauge condition.

Locally, gauge fixing conditions, $C^a = 0$, which really serve as additional constraints, are generally required to satisfy two criteria:

i) given any set of canonical variables, there must exist a gauge transformation which brings it into the chosen gauge, and
ii) the chosen conditions must fix the gauge completely.

Together, these two criteria imply that there must be just as many gauge conditions as there are first class constraints, and that the commutators

$$\{C^a, \phi_b\}$$

form an invertible matrix, where ϕ_b are the original first class constraints, by this construction turned into second class constraints.

Some authors define a number of additional Hamiltonians. Typically, the first and second class constraints are separated, with the multiplier conditions for the latter being incorporated back into the Hamiltonian. As I will illustrate in Example 3, this can lead to changes, though they are "unphysical," in the value of the constrained degrees of freedom. Another modification is employed in the first class case (since those of interest are generally gauge theories), if there are additional such constraints which were not present in the primary Hamiltonian: this modification results in extra Lagrange multipliers being introduced for the subsidiary first class constraints, leading to an 'extended' Hamiltonian. I prefer to understand the situation, in the first place, without these additional measures, since they depend for their validity, on further information about the true nature of the physical system being considered. But for typical physical applications they may be essential.

Before going on to consider specific examples, several further comments are in order. First, an important point which is often neglected is that for the various definitions to be workable, the constraints must satisfy certain regularity properties. These can be best exemplified by saying that, in any variation for which δq and δp are $O(\epsilon)$, then $\delta\phi(q,p)$ must also be $O(\epsilon)$. Thus of

$$p = 0, \quad p^2 = 0, \quad \sqrt{p} = 0,$$

only the first is acceptable, while

$$p_1^2 + p_2^2 = 0, \text{ which implies } \ p_1 = 0, \quad p_2 = 0,$$

only the latter two constraints are acceptable. Finally, none of this at all is necessary if we simply wish to know a set of equations to solve for the dynamics. However, it does serve a purpose if we wish to know how to remove non-dynamical degrees of freedom from the formulation. This, in particular, may be essential if we are to avoid quantizing non-dynamical aspects of the problem, and is the main area where current interest in the subject lies.

Illustrative examples

Example 1

This first example stresses that any quantity depending upon $2N$ degrees of freedom cannot arbitrarily be treated as a canonical Hamiltonian. In fact, this example also deals with a situation in which a purported Hamiltonian has an odd number of arguments, a circumstance which arises frequently for coherent states based on generalized group representations. A canonical Hamiltonian is properly defined as a function(al) of paired sets of variables, the coordinates of phase space which generate its symplectic geometry. Thus the function

$$F(a,b,c) = V(c) + \frac{1}{2}(a^2 + b^2) \ ,$$

is not, strictly speaking, a Hamiltonian, since no pairing of the variables is indicated which would require them to satisfy the Possion bracket relation $\{q,p\} = 1$. In the way by which interest in this particular problem arises, its author was really looking at a situation which can be best described by a Lagrangian in a non-traditional first order form:

$$\mathcal{L} = \dot{a}\cos c + \dot{b}\sin c - V(c) - \frac{1}{2}(a^2 + b^2) \ .$$

It is obvious that if we now try to identify

$$p_a = \cos c, \quad p_b = \sin c, \quad p_c = 0 \ ,$$

then each of these must be regarded as giving rise to a separate constraint:

$$\begin{aligned} \phi^0 &= p_a - \cos c \\ \phi^1 &= p_b - \sin c \\ \phi^2 &= p_c \end{aligned}$$

Thus, the primary Hamiltonian becomes:

$$\mathcal{H}_p = V(c) + \frac{1}{2}(a^2 + b^2) + \lambda_0(p_a - \cos c) + \lambda_1(p_b - \sin c) + \lambda_2 p_c \ .$$

As previously, we now look to see if their are any additional constraints:

$$\begin{aligned} \dot{\phi}^0 &= \{\phi^0, \mathcal{H}_p\} = -a + \sin c\lambda_2 \\ \dot{\phi}^1 &= \{\phi^1, \mathcal{H}_p\} = -b - \cos c\lambda_2 \\ \dot{\phi}^2 &= \{\phi^2, \mathcal{H}_p\} = -\left\{\frac{\partial V}{\partial c} + \lambda_0 \sin c - \lambda_1 \cos c\right\} \end{aligned}$$

These give the one additional constraint

$$\psi = a\cos c + b\sin c \ ,$$

for which we find

$$\dot{\psi} = \{\psi, \mathcal{H}_p\} = \lambda_0 \cos c + \lambda_1 \sin c + \lambda_2(-a \ \sin c + b \ \cos c) \ .$$

Now we have a set of equations sufficient to find all the λ's. As none of the λ's remain undetermined, all the constraints will turn out to be second class. Thus only two dynamical variables will remain, corresponding to one unconstrained canonical degree of freedom. We can tell that there is only one unconstrained canonical degree of freedom in the problem, without having found it explicitly at this stage.

In this problem, the constrained and unconstrained degrees of freedom can be decoupled. To carry that out it is helpful to proceed in several discrete steps. We consider, then, the set of variables

$$
\begin{aligned}
&A = a\ \cos c + b\ \sin c = \psi^1 \approx 0\ ,\\
&P_A = p_a\ \cos c + p_b\ \sin c - 1 = \phi^0\ \cos c + \phi^1\ \sin c = \tilde{\phi}^0 \approx 0\\
&\tilde{B} = -a\ \sin c + b\ \cos c\ ,\\
&P_B = -p_a\ \sin c + p_b\ \cos c = -\phi^0\ \sin c + \phi^1\ \cos c = \tilde{\phi}^1 \approx 0\\
&C = c\ \ ,\ \ p_c = \phi^2 \approx 0\ .
\end{aligned}
$$

Note

$$
\begin{aligned}
\{A, p_c\} = \tilde{B}\ \ ,&\quad \{P_A, p_c\} = P_B\\
\{\tilde{B}, p_c\} = -A\ \ ,&\quad \{P_B, p_c\} = -P_A - 1
\end{aligned}
$$

Finally, we introduce:

$$
P_C = p_c - \tilde{B}(P_A + 1) + AP_B\ ,
$$

and

$$
B = \tilde{B} + P_C = \phi^2 - \tilde{B}\tilde{\phi}^0 + \psi^1\tilde{\phi}^1\ ,
$$

which has become a constraint replacing p_c. Now we have $A \approx 0,\ P_A \approx 0,\ B \approx 0$ & $P_B \approx 0$ as second class constraints, while C & P_C survive as an unconstrained canonical pair; i.e. we have conveniently arranged our degrees of freedom into two canonical pairs composed entirely of constraints, and one remaining pair representing the only true dynamical degree of freedom in the problem.

A judicious change of variables at the beginning (and, perhaps, with hindsight) can greatly simplify the ensuring analysis. Since it will provide us with material for a later example, we now carry this out. With the new definitions

$$
\begin{aligned}
A &= a\ \cos c + b\ \sin c\\
B &= -a\ \sin c + b\ \cos c
\end{aligned}
$$

we find that the Lagrangian can be rewritten as

$$
\mathcal{L} = \dot{A} - B\dot{C} - V(c) - \frac{1}{2}(A^2 + B^2)
$$

in which the 'A' degree of freedom decouples completely, and the further identification $P_c \equiv -B$ even eliminates the remaining pair of second class constraints, bringing us to the point we finished with above regarding the single 'true' dynamical degree of freedom.

Example 2

This second example shows that sometimes, whether constraints are first or second class can depend on properties of the Hamiltonian. We shall also use this example to give a specific illustration of Dirac's procedure. the example is provided by the Lagrangian:

$$\mathcal{L} = \frac{1}{2} e^{y} \dot{x}^2 - V(x,y) \ .$$

It is clear that in this case there is one primary constraint.

$$\phi^0 = p_y \ ,$$

so that the primary Hamiltonian becomes

$$\mathcal{H}_p = \frac{1}{2} e^{-y} p_x^2 + V(x,y) + \lambda p_y$$

For the time derivative of ϕ^0 we find

$$\dot{\phi}^0 = \{\phi^0, \mathcal{H}\} = \frac{1}{2} e^{-y} p_x^2 - \frac{\partial V}{\partial y}$$

indicating that we must introduce an additional constraint (which we assume to be):

$$\phi^1 = p_x - f(x,y), \quad \text{where} \quad f^2(x,y) = 2e^y \frac{\partial V(x,y)}{\partial y} \ .$$

The time derivative of ϕ^1 leads to an equation for λ.

$$\dot{\phi}^1 = \{\phi^1, \mathcal{H}\} = -\frac{\partial V}{\partial x} + \lambda \frac{\partial f}{\partial y} \approx 0 \ ,$$

while the Possion bracket $\{\phi^0, \phi^1\}$ does not vanish, so these contraints are second class.

Now, one reason why this example is interesting is that without specific knowledge of $V(x,y)$, we cannot explicitly decouple the constraints from the dynamical degrees of freedom. This is where Dirac's procedure comes to our rescue. Using the definition given above, note that we here have:

$$\begin{aligned} \{x, p_x\}_{DB} = \{x, p_x\} &- \{x, \phi^0\}\{\phi^0, \phi^1\}^{-1}\{\phi^1 p_x\} \\ &- \{x, \phi^1\}\{\phi^1, \phi^0\}^{-1}\{\phi^0 p_x\} = 1 \ , \end{aligned}$$

as does $\{x, f(x,y)\}_{DB}$, so in particular we now have $\{x, y\}_{DB} \neq 0$, because via ϕ^1, y is no longer independent of p_x and x.

Example 2a

A second reason why the above example is so interesting is that it seems to take on an entirely different character when the potential $V(x,y)$ vanishes, since the constraints then become equivalent to

$$\phi^0 = p_y, \quad \phi^1 = p_x$$

and they are now both first class. The solution for the dynamics is given by

$$x = \text{constant}, \;\; y = \text{any function of time.}$$

This solution can certainly be well understood, but the problem itself does not fit easily into the mold people have tried to cut out for it, since the general inclination to interpret first class constraints as generates of gauge transformations clearly is inappropriate for ϕ^1.

Example 3

The third example is derived from our first, by taking from it the decoupled constrained degree of freedom, A, alone, and shows that solving and substituting for the Lagrange multipliers can change the value of the constrained variables. We take for the Lagrangian:

$$\mathcal{L} = \dot{A} - \frac{1}{2}A^2.$$

The details of the calculation can be done as an exercise, the results of which are easily summarized. The constraints are found to be

$$P_A - 1 = 0 \text{ and } A = 0 \ ,$$

and the solution for the Lagrange multiplier is $\lambda = 0$. However, if we substitute this back into the Hamiltonian, the subsequent solution for the constrained variables becomes:

$$A = c_1 \text{ and } P_A = c_1 T + c_2 \ ,$$

which is very different, **unless we use the original solution to the constraints as part of the initial data!** Frequently, in the treatment of gauge theories for physical systems, differences like this are often treated as irrelevant, but one may sometimes be too cavalier in dismissing such changes to the theory, especially if one has quantization in mind.

Example 4

For the final example, I refer again to the original example of electromagnetism. It is clear there that taking the additional constraints (gauge fixing conditions), $A_0 = 0$ and $A = 0$, gives us a situation with entirely second class constraints, and completely fixes the gauge. On the other hand the condition $\partial_\mu A^\mu = 0$ does not: it is not enough conditions, and is insensitive to the change $A_\mu \to A_\mu + \partial_\mu \phi$, for any ϕ which satisfies $\nabla^2 \phi = 0$.