I. BEHAVIOUR AND SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

Suppose we have a general second order operator

$$\left(\frac{d^2}{dx^2} + p\frac{d}{dx} + q\right) y = 0 \ . \tag{1.1}$$

Let us substitute the following

$$y = e^{\int f(x)dx} z \ . \tag{1.2}$$

One finds (prime denoting $\frac{d}{dx}$)

$$\frac{y'}{y} = f + \frac{z'}{z} \tag{1.3}$$

$$\frac{y^{(n+1)}}{y} = \left(\frac{y^{(n)}}{y}\right)' + \frac{y^{(n)}}{y}\frac{y'}{y} \tag{1.4}$$

Thus, we get

$$\frac{y''}{y} = f' + \frac{z''}{z} + f^2 + 2f\frac{z'}{z} \tag{1.5}$$

and the main result

$$z'' + (p + 2f)z' + (q + f' + pf + f^2)z = 0 \ . \tag{1.6}$$

Now let us put in the condition

$$f = -\frac{1}{2}p \tag{1.7}$$

so that

$$z'' + \left(q - \frac{1}{2}p' - \frac{1}{4}p^2\right) z = 0 \tag{1.8}$$

and we get the invariant

$$Q = q - \frac{1}{2}p' - \frac{1}{4}p^2 \tag{1.9}$$

of the equivalence class of equations.

Let us do the following change, $x \to r(x)$, such that $z'' + Qz = 0$ becomes ($\dot{}$ denotes $\frac{d}{dr}$)

$$(r')^2\ddot{z} + r''\dot{z} + Qz = 0 \ . \tag{1.10}$$

(The domains over which solutions can be written may be quite different; i.e. $(-\pi,\pi) \to (-\infty,\infty)$.) Now, Eq. (1.10) can be rewritten as

$$(r')^2\left(\ddot{z} + \frac{r''}{(r')^2}\dot{z} + \frac{Q}{(r')^2}z\right) = 0 \ . \tag{1.11}$$

Noting that

$$\frac{d}{dr} = \frac{1}{r'}\frac{d}{dx} \tag{1.12}$$

we can write

$$\begin{aligned} -\frac{1}{2}\frac{1}{r'}\left(\frac{r'''}{(r')^2} - \frac{2(r'')^2}{(r')^3}\right) - \frac{1}{4}\frac{(r'')^2}{(r')^4} &= -\frac{1}{2(r')^2}\left(\frac{r'''}{r'} - \frac{3}{2}\frac{(r'')^2}{(r')^2}\right) \\ &= \frac{\left((r')^{-\frac{1}{2}}\right)''}{(r')^2(r')^{-\frac{1}{2}}} \ . \end{aligned} \tag{1.13}$$

Using Eq. (1.13) and the transformation $z \to z e^{\frac{1}{2}\int \frac{r''}{r'}dx}$ we get

$$\left(\frac{d^2}{dr^2} + \frac{1}{(r')^2}\left(Q + \frac{\left((r')^{-\frac{1}{2}}\right)''}{(r')^{-\frac{1}{2}}}\right)\right) z e^{\frac{1}{2}\int \frac{r''}{r'}dx} = 0 \ . \tag{1.14}$$

(The Schwarzian derivative is given by

$$\{r, x\} \equiv \left(\frac{r'''}{r'} - \frac{3}{2}\frac{(r'')^2}{(r')^2}\right) , \tag{1.15a}$$

$$\{x, r\} = \frac{1}{(r')^2}\{r, x\} \ . \tag{1.15b}$$

For some function S

$$\{S, x\} \equiv \{S, r\}(r')^2 + \{r, x\} \ .) \tag{1.16}$$

Note that

$$e^{\frac{1}{2}\int \frac{r''}{r'}dx} = e^{(\ln \frac{r''}{r'})\frac{1}{2}} = \sqrt{r'} \ . \tag{1.17}$$

Thus Eq. (1.14) becomes

$$\left(\frac{d^2}{dr^2} + \frac{1}{(r')^2}\left(Q + \frac{\left((r')^{-\frac{1}{2}}\right)''}{(r')^{-\frac{1}{2}}}\right)\right)\sqrt{r'}z = 0 \ . \tag{1.18}$$

Suppose that r is chosen such that the second term of Eq. (1.18) is zero, then a solution z of Eq. (1.18) is

$$z = (r')^{-\frac{1}{2}} \ . \tag{1.19}$$

A linearly independent solution of $\frac{d^2}{dr^2}\sqrt{r'}z = 0$ is

$$z = r(r')^{-\frac{1}{2}} \ . \tag{1.20}$$

Now suppose that the second term of Eq. (1.18) is equal to one, then we find for

$$\left(\frac{d^2}{dr^2} + 1\right)\sqrt{r'}z = 0$$

that

$$z = \frac{e^{\pm ir}}{\sqrt{r'}} \ . \tag{1.21}$$

Eq. (1.21) is similar in form to a solution like

$$G = \frac{e^{\pm\int\sqrt{F}dx}}{F^{\frac{1}{4}}} \ . \tag{1.22}$$

One uses this in a W.K.B. approximation. Every solution can be written in the form Eq. (1.22). As an example we can write

$$\frac{\sin r}{\sqrt{r'}} = \frac{e^{ig}}{\sqrt{g'}} \ . \tag{1.23}$$

This g must be a non-trivial complex function.

II. FORM OF SOLUTIONS NEAR REGULAR SINGULAR POINTS

For

$$z'' + Qz = 0 \ , \tag{2.1}$$

if we have a solution z_1 we can always find another solution

$$z_2 = z_1 \int \frac{1}{(z_1)^2} dx \ . \tag{2.2}$$

For

$$y'' + py' + qy' = 0 \ , \tag{2.3}$$

we have the analogous second solution

$$y_2 = y_1 \int \frac{W(x)}{(y_1)^2} dx \ . \tag{2.4}$$

Consider the following (form of the confluent hypergeometric) equation

$$z'' + \left(-\alpha^2 + \frac{2\alpha\beta}{x} + \frac{\frac{1}{4} - \gamma^2}{x^2} \right) z = 0 \ . \tag{2.5}$$

This has a regular singular point at $x = 0$ and an irregular singular point at $x = \infty$. Now, for $x \to 0$, z will have the behaviour

$$z \to x^{\frac{1}{2} + \epsilon' \gamma}, \ \epsilon' = \pm 1 \tag{2.6a}$$

and for $x \to \infty$, z will have the behaviour

$$z \to e^{\epsilon \alpha x} x^{-\epsilon \beta}, \ \epsilon = \pm 1 \ . \tag{2.6b}$$

For an arbitrary point (ordinary) we can impose whatever boundary conditions we want

$$x \to x_0 : \ z = 1, \ (x - x_0) \ .$$

For the hypergeometric equation we would consider

$$x(x-1)z'' + \left((\frac{1}{4} - \alpha^2) - \frac{(\frac{1}{4} - \beta^2)}{x} + \frac{(\frac{1}{4} - \gamma^2)}{x - 1} \right) z = 0. \tag{2.7}$$

The behaviour near $x \to 1, 0, \infty$ will be

$$x \to 1 : (x-1)^{\frac{1}{2} + \epsilon'' \gamma}, \epsilon'' = \pm 1 \tag{2.8a}$$

$$x \to 0 : x^{\frac{1}{2}+\epsilon'\beta}, \epsilon' = \pm 1 \tag{2.8b}$$

$$x \to \infty : x^{\frac{1}{2}+\epsilon\alpha}, \epsilon = \pm 1. \tag{2.8c}$$

Now let us look at solving equations like Eq. (2.5). Suppose we factor out the quantity

$$z = x^{\frac{1}{2}+\epsilon'\gamma} e^{\epsilon\alpha x} u(x) \tag{2.9}$$

where $u(x)$ is some series. We will find that if

$$\frac{1}{2} + \epsilon'\gamma + N \equiv -\epsilon\beta \tag{2.10}$$

then the series will truncate. It is most important (necessary) for truncation that N be an integer. Putting Eq. (2.9) in Eq. (2.5) we get

$$u'' + 2\left(\epsilon\alpha + \frac{\frac{1}{2} + \epsilon'\gamma}{x}\right)u' + \frac{2\epsilon\alpha(\epsilon\beta + \frac{1}{2} + \epsilon'\gamma)}{x} u = 0. \tag{2.11}$$

Now let

$$u = \sum_n a_n x^n \tag{2.12}$$

so that we get the following recurrence relation

$$(n+1)na_{n+1} + 2\epsilon\alpha(\frac{1}{2} + \epsilon\beta + \epsilon'\gamma)a_n + (\frac{1}{2} + \epsilon'\gamma)(n+1)a_{n+1} + 2\epsilon\alpha n a_n = 0. \tag{2.13}$$

Truncation occurs if there exists an N such that

$$a_{n+1} \equiv 0 \tag{2.14}$$

which is implied by Eq. (2.10).

Let us operate on $x*$(Eq. (2.11)) n times with $\frac{d}{dx}$. We will get

$$\left(\left(x\frac{d^2}{dx^2} + 2\left(\epsilon\alpha x + \frac{1}{2} + \epsilon'\gamma + \frac{1}{2}n\right)\frac{d}{dx} + 2\epsilon\alpha\left(\epsilon\beta + \frac{1}{2} + \epsilon'\gamma\right) + 2\epsilon\alpha n\right)\frac{d^n}{dx^n}\right)u = 0 \tag{2.15}$$

Note that this equation is always second order in the n^{th} derivative, and that when $\epsilon\beta + \frac{1}{2} + \epsilon'\gamma$ is a negative integer $-N$, then one solution for $\frac{d^N u}{dx^N}$ is a constant (i.e., gives the truncating solution). Also, from Eq. (2.15) we see that $\frac{d^n}{dx^n}$ acting on u is a shift (cf. raising or lowering) operator, with the action

$$\gamma \to \epsilon'\gamma + \frac{\epsilon' n}{2}, \ \beta \to \epsilon'\beta + \frac{\epsilon n}{2}. \tag{2.16}$$

In Quantum Mechanics, Eq. (2.7) arises for the (spin-weighted) spherical harmonics with

$$\alpha = \ell + \frac{1}{2}$$
$$\beta = \frac{1}{2}(s - m)$$
$$\gamma = \frac{1}{2}(s + m) ,$$

and Eq. (2.5) arises for the Coulomb (radial) wave functions, with bound state occurence being given by a condition for truncating solutions.

From Eq. (2.7) lets now pull out the factor

$$z = x^{\frac{1}{2}+\epsilon'\beta}(x-1)^{\frac{1}{2}+\epsilon''\gamma}u(x) \tag{2.17}$$

then (after several lines of algebra) we will find the condition for truncation is:

$$\frac{1}{2} + \epsilon'\beta + \frac{1}{2} + \epsilon''\gamma + N = \frac{1}{2} + \epsilon\alpha. \tag{2.18}$$

In general, we find that $\frac{d^k u}{dx^k}$ gives rise to a solution of Eq. (2.5) with $\beta \to \beta + \epsilon'\frac{k}{2}; \gamma \to \gamma + \epsilon''\frac{k}{2}$.

Raising and lowering operator characteristics are determined entirely by techniques considered above.

III. RECURRENCE RELATIONS

In general, a recurrence relation has the form

$$\tilde{y} = \alpha(x)y + \beta(x)\frac{\partial y}{\partial x}. \tag{3.1}$$

We want to devise a method, in principle, for understanding why recurrence relations exist and what characterizes their behaviour.

Rewrite Eq. (3.1) as

$$\tilde{y} = \beta\left(\frac{\partial}{\partial x} + \frac{\alpha}{\beta}\right) y \ , \tag{3.2}$$

or alternatively

$$\tilde{y} = B\frac{\partial}{\partial x}Ay \ . \tag{3.3}$$

In the simplest recurrence relation for Eq. (2.5) above we would have

$$\frac{\alpha}{\beta} \sim \mu + \frac{\lambda}{x} \tag{3.4a}$$

and for Eq. (2.7)

$$\frac{\alpha}{\beta} \sim \frac{\sigma}{x} + \frac{\tau}{x-1} \ . \tag{3.4b}$$

Then we would have

$$A(x) = e^{\mu x}x^{\lambda} \text{ or } x^{\sigma}(x-1)^{\tau} \tag{3.5}$$

as given by the constructions indicated in the previous section.

Consider the two equations

$$(\partial_{xx} + q(x))y = 0 \tag{3.6a}$$

$$(\partial_{xx} + Q(x))z = 0 \ . \tag{3.6b}$$

We want to seek α, β (which truncate) such that

$$y = \alpha z + \beta z' \ . \tag{3.7}$$

Let Eq. (3.6a–b) be anything we wish to write down. Suppose that two linearly independent solutions y_1, y_2 exist for Eq. (3.6a). Also, let the same be true for z_1, z_2 of Eq. (3.6b). A possible, but not useful, situation is when Eq. (3.7) would map $z_1 \to y_1$ but would permit $z_2 \to$ anything, e.g.,

$$y_1 = \left(f(x)\left(\partial_x - \frac{z_1'}{z_1}\right) + \frac{y_1}{z_1}\right)z_1 \ . \tag{3.8}$$

However, $f(x)$ has no restrictions so that Eq. (3.8) is not very useful – it is too general.

Consider the equations

$$y_1 = \alpha z_1 + \beta z_1' \tag{3.9a}$$

$$y_2 = \alpha z_2 + \beta z_2' \ . \tag{3.9b}$$

The solutions for α and β are

$$\alpha = \frac{-z_1' y_2 - z_2' y_1}{W(z_1, z_2)} \tag{3.10a}$$

$$\beta = \frac{z_1 y_2 - z_2 y_1}{W(z_1, z_2)} \ . \tag{3.10b}$$

where $W(\, , \,)$ is the Wronskian. In general there exists an inverse map

$$z = \frac{1}{k}((\alpha + \beta')y - \beta y') \tag{3.11}$$

where

$$k = \frac{W(y_1, y_2)}{W(z_1, a_2)} = \text{constant} \ . \tag{3.12}$$

Thus our mapping and its inverse exists and is unique for the Eqs. (3.6a,b). Now

$$(\partial_{xx} + q(x)) = \left(\partial_x + \frac{\alpha + \beta'}{\beta}\right)\left(\partial_x - \frac{\alpha + \beta'}{\beta}\right) + \frac{k}{\beta^2} \tag{3.13a}$$

$$(\partial_{xx} + Q(x)) = \left(\partial_x - \frac{\alpha}{\beta}\right)\left(\partial_x + \frac{\alpha}{\beta}\right) + \frac{k}{\beta^2} \ . \tag{3.13b}$$

Thus

$$q = \frac{k}{\beta^2} - \left(\frac{\alpha + \beta'}{\beta}\right) - \left(\frac{\alpha + \beta'}{\beta}\right)^2 \tag{3.14a}$$

$$Q = \frac{k}{\beta^2} + \left(\frac{\alpha}{\beta}\right)' - \left(\frac{\alpha}{\beta}\right)^2 \ . \tag{3.14b}$$

When β is a constant we get the simplest examples of "raising" and "lowering" type of operators.

If we choose our mapping as

$$z_1 \to a y_1 + b y_2 \tag{3.15a}$$

$$z_2 \to c y_1 + d y_2 \tag{3.15b}$$

such that $ad - bc \neq 0$, then for $b = c = 1$, $a = d = 0$,

$$\beta = \frac{y_1 z_1 - y_2 z_2}{W} \ . \tag{3.16}$$

We find that there are four linearly independent β's. It is not always clear, however, which β to pick, or that any useful choice is available.

Look at

$$\beta^2(\partial_{xx} + q) = (\beta\partial_x + \alpha)(\beta\partial_x - \alpha - \beta') + k \tag{3.17a}$$

$$\beta^2(\partial_{xx} + Q) = (\beta\partial_x - \alpha - \beta')(\beta\partial_x + \alpha) + k \tag{3.17b}$$

where for Eqs. (3.17a,b) define

$$A = (\beta\partial_x + \alpha) \tag{3.18a}$$

$$B = (\beta\partial_x - \alpha - \beta') \tag{3.18b}$$

so that

$$[A, B] \neq 0 \ . \tag{3.19}$$

Notice that

$$B(AB + k) = (BA + k)B \tag{3.20a}$$

$$A(BA + k) = (AB + k)B \ . \tag{3.20b}$$

Note that, for the operators multiplied by β^2, k simply represents a shift in the eigenvalues (spectrum) of the operators AB and BA: this β^2 has really changed us to quite new operators and knowing the zeros of β is very important.

For simplicity, consider a case of the whole interval $(-\infty, \infty)$. Suppose that the behaviour of y in Eq. (3.6a) is

$$y_1 = \lim_{x\to-\infty} e^{i\omega x} = \lim_{x\to-\infty}(Ae^{i\omega x} + Be^{-i\omega x}) \tag{0.1a}$$

$$y_2 = \lim_{x\to-\infty} e^{-i\omega x} = \lim_{x\to-\infty}(Ce^{i\omega x} + De^{-i\omega x}) \ , \tag{0.1b}$$

and similarly for z in Eq. (3.6b). Then β will have the form as $x \to -\infty$

$$\begin{aligned} \beta &= ay_1z_1 + by_1z_2 + cy_2z_1 + dy_2y_2 \\ &\simeq e^{-2i\omega x} + \text{const} + \text{const} + e^{2i\omega x} \ . \end{aligned} \tag{0.2}$$

To avoid a β which oscillates, one would choose $a = d = 0$. Now, as $x \to \infty$, β will have the form

$$\beta = b(Ae^{i\omega x} + Be^{-i\omega x})(\tilde{C}e^{i\omega x} + \tilde{D}e^{-i\omega x}) + c(Ce^{i\omega x} + De^{-i\omega x})(\tilde{A}e^{i\omega x} + \tilde{B}e^{-i\omega x}) \tag{0.3}$$

where the tildes are associated with solutions z_1, z_2 of z for Eq. (3.6b). Similarly, we require, at $\pm\infty$,

$$bA\tilde{C} + cC\tilde{A} = 0 \ , \tag{0.3a}$$

$$bB\tilde{D} + cD\tilde{B} = 0 \ . \tag{0.3b}$$

This keeps our solutions from oscillating at $x \to \pm\infty$. Therefore

$$\frac{b}{c} = -\frac{C\tilde{A}}{A\tilde{C}} = -\frac{D\tilde{B}}{B\tilde{D}} \tag{0.4}$$

and

$$\frac{CB}{AD} = \frac{\tilde{C}\tilde{B}}{\tilde{A}\tilde{D}} \ . \tag{0.5}$$

Looking at

$$1 - \frac{CB}{AD} = \frac{1}{AD} = \frac{1}{\tilde{A}\tilde{D}} \tag{0.6}$$

where, in this case, $k = 1$. This implies

$$AD = \tilde{A}\tilde{D} \tag{0.7}$$

$$BC = \tilde{B}\tilde{C} \ . \tag{0.8}$$

These conditions help to imply that $A, \tilde{A}$ must have the same zeros (poles) and $B, \tilde{B}$ have the same zeros (poles); i.e., that the singular points of the scattering data correspond. Otherwise, b or c would be zero and this would mean that z_1, z_2 would map to the same function y. Thus, for β to be non-oscillatory (it is useless otherwise) we must have, at least, that the singular points of the scattering data correspond. This is in fact a very restrictive condition on q, Q in Eq. (3.6a ,3.6b), so that useful recurrence relations can occur only in special situations. However, precisely these situations can arise in the use of inverse scattering techniques to solve non-linear (completely integrable) evolution equations, which can give multi-parameter potentials (*i.e.,* q, Q) for linear equations with trivially related scattering data.

Finally, recall Eq. (3.13b)

$$(\partial_{xx} + Q(x)) = \left(\partial_x - \frac{\alpha}{\beta}\right)\left(\partial_x + \frac{\alpha}{\beta}\right) + \frac{k}{\beta^2} \ .$$

When $k = 0$ we know we have a solution. Once we have this solution we can get all the solutions for y and z. Similarly, once we have only <u>two</u> solutions β, we can solve for everything (even though β can be shown to satisfy a fourth order differential equation!).