9/10/18
R.P. Woodard



## Solutions to Exam #1

**1a** * Is this a joke? $m = \frac{m_1 m_2}{m_1+m_2} = \frac{2}{9}M$

**1b** * $L = \frac{1}{2} m(\dot r^2 + r^2\dot\phi^2) - \frac{1}{2} k r^2$

* $m r^2 \dot\phi = L \rightarrow \dot\phi = \frac{L}{m r^2}$

* $\frac{1}{2} m \dot r^2 + \frac{L^2}{2 m r^2} + \frac{1}{2} k r^2 = E \rightarrow \dot r = \pm\sqrt{\frac{2E}{m} - \frac{L^2}{m^2 r^2} - \frac{k}{m} r^2}$

* $r' = \dot r/\dot\phi = \pm r^2 \sqrt{\frac{2mE}{L^2} - \frac{mk}{L^2} r^2 - \frac{1}{r^2}} = \pm r^3 \sqrt{-\frac{1}{r^4} + \frac{2mE}{L^2}\frac{1}{r^2} - \frac{mk}{L^2}}$

**1c** * $\dot r = 0 \rightarrow \frac{2E}{m} - \frac{L^2}{m^2 r^2} - \frac{k}{m} r^2 = 0 \rightarrow r_\pm^2 = \frac{E}{k}\left[1 \pm \sqrt{1 - \frac{kL^2}{mE^2}}\right]$

**1d** * $\frac{1}{r^3} r' = -\frac{1}{2} v' = \pm\sqrt{-v^2 + \frac{2mE}{L^2} v - \frac{mk}{L^2}} \rightarrow d\phi = \frac{dv}{2\sqrt{-v^2 + \frac{2mE}{L^2} v - \frac{mk}{L^2}}} = \frac{1}{2} d \sin^{-1}\left(\frac{v - mE/L^2}{\sqrt{\frac{m^2E^2}{L^4} - \frac{mk}{L^2}}}\right)$

* $v(\phi) = \frac{mE}{L^2}\left[1 + \sqrt{1 - \frac{kL^2}{mE^2}} \sin[2(\phi - \phi')]\right] \rightarrow r(\phi) = \frac{L/\sqrt{mE}}{\sqrt{1 + \sqrt{1 - \frac{kL^2}{mE^2}} \sin[2(\phi - \phi')]}}$

* NB $r(\phi)$ goes from $r_+$ to $r_-$ twice in one orbit

**1e** * $\frac{1}{4} T = \int_{r_-}^{r_+} \frac{dr}{\sqrt{\frac{2E}{m} - \frac{L^2}{m^2 r^2} - \frac{k}{m} r^2}}$ (Note the factor of $\frac{1}{4} \neq \frac{1}{2}$)

**2a** * $M = \rho_0 \int_{-b}^{b} dz \int_0^{a\sqrt{1-z^2/b^2}} d\rho\, \rho \int_0^{2\pi} d\phi = 2\pi\rho_0 \int_0^b dz\, a^2\left[1 - \left(\frac{z}{b}\right)^2\right] = \frac{4\pi}{3} a^2 b \rho_0$

* $\vec R = \rho_0 \int_{-b}^{b} dz \int_0^{a\sqrt{1-z^2/b^2}} d\rho\, \rho \int_0^{2\pi} d\phi\, [\rho\hat\rho + z\hat z] = 0$

**2b** * $I_{ij} = \rho_0 \int_{-b}^{b} dz \int_0^{a\sqrt{1-z^2/b^2}} d\rho\, \rho \int_0^{2\pi} d\phi \left[\delta_{ij}(\rho^2 + z^2) - (\rho\hat\rho_i + z\hat z_i)(\rho\hat\rho_j + z\hat z_j)\right] = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$

* $I_{11} = I_{22} = 2\pi\rho_0 \int_{-b}^{b} dz \int_0^{a\sqrt{1-z^2/b^2}} d\rho\, \rho \left[\frac{1}{2}\rho^2 + z^2\right] = 4\pi\rho_0 \int_0^b dz \left[\frac{1}{8} a^4 \left(1 - \frac{z^2}{b^2}\right)^2 + \frac{1}{2} a^2 \left(1 - \frac{z^2}{b^2}\right) z^2\right] = M\left(\frac{a^2+b^2}{5}\right)$

* $I_{33} = 2\pi\rho_0 \int_{-b}^{b} dz \int_0^{a\sqrt{1-z^2/b^2}} d\rho\, \rho^3 = 4\pi\rho_0 \int_0^b dz\, \frac{1}{4} a^4 \left(1 - 2\frac{z^2}{b^2} + \frac{z^4}{b^4}\right) = \frac{4\pi}{3}\rho_0 a^2 b \times \frac{2}{5} a^2 = M \times \frac{2}{5} a^2$

**2c** * $\hat x$ has $I_{11}$

* $\hat y$ has $I_{11}$

* $\hat z$ has $I_{33}$

$\Omega = \left(\frac{a^2 - b^2}{a^2 + b^2}\right)\omega$

**2d** 
$$\left.\begin{aligned} &* \dot\omega_1 = +\left(1 - \frac{I_{33}}{I_{11}}\right)\omega\,\omega_2 \\ &* \dot\omega_2 = -\left(1 - \frac{I_{33}}{I_{11}}\right)\omega\,\omega_1 \end{aligned}\right\} \rightarrow \left\{\begin{aligned} \omega_1(t) &= \omega_1(0)\cos(\Omega t) - \omega_2(0)\sin(\Omega t) \\ \omega_2(t) &= +\omega_1(0)\sin(\Omega t) + \omega_2(0)\cos(\Omega t) \end{aligned}\right\}$$

**2e** * Is this another joke?

$L_i = I_{ij}\omega_j \rightarrow L_1 = I_{11}\omega_1(t)$, $L_2 = I_{11}\omega_2(t)$ & $L_3 = I_{33}\omega$

Solutions to Exam #1

(3a) $$L = \tfrac{1}{2} m_1 \dot{d}_1^2 + \tfrac{1}{2} m_2 \dot{d}_2^2 + m_1 g d_1 + m_2 g d_2 - \tfrac{1}{2} k_1 (d_1 - \ell_1)^2 - \tfrac{1}{2} k_2 (d_2 - d_1 - \ell_2)^2$$

(3b) $$\left.\begin{aligned} &* \frac{\delta S}{\delta d_1} = 0 \Rightarrow m_1 \ddot{d}_1 = m_1 g - k_1 (d_1 - \ell_1) + k_2 (d_2 - d_1 - \ell_2) \\ &* \frac{\delta S}{\delta d_2} = 0 \Rightarrow m_2 \ddot{d}_2 = m_2 g - k_2 (d_2 - d_1 - \ell_2) \end{aligned}\right\} \text{both} = 0 \text{ at equilib}$$

$$\therefore \; \bar{d}_1 = \ell_1 + \left(\frac{m_1 + m_2}{k_1}\right) g \quad \& \quad \bar{d}_2 = \ell_1 + \ell_2 + \left(\frac{m_1 + m_2}{k_1}\right) g + \frac{m_2}{k_2} g$$

(3c) * NB if we shift by these values $\Delta d_i = d_i - \bar{d}_i$

$$\Rightarrow \begin{pmatrix} m_1 \Delta \ddot{d}_1 \\ m_2 \Delta \ddot{d}_2 \end{pmatrix} = - \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} \Delta d_1 \\ \Delta d_2 \end{pmatrix} \Rightarrow \Omega^2 = \begin{pmatrix} \frac{k_1 + k_2}{m_1} & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & k_2/m_2 \end{pmatrix}$$

$$* \det[\Omega^2 - \omega^2 I] = 0 \Rightarrow \boxed{\omega_\pm^2 = \frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2} \pm \sqrt{\left(\frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2}\right)^2 - \frac{k_1 k_2}{m_1 m_2}}}$$

* NB we can also write $\omega_\pm^2$ as $\frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2} \pm \sqrt{\left(\frac{k_1 - k_2}{2m_1} - \frac{k_2}{2m_2}\right)^2 + \frac{k_1 k_2}{m_1^2}}$
which establishes that it is real

(3d) $$(\Omega^2) \begin{pmatrix} \Delta d_1 \\ \Delta d_2 \end{pmatrix} = \omega_\pm^2 \begin{pmatrix} \Delta d_1 \\ \Delta d_2 \end{pmatrix} \Rightarrow \Delta d_2 = \left[\frac{k_1 + k_2}{2m_1} - \frac{k_2}{2m_2} \mp \sqrt{\left(\frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2}\right)^2 - \frac{k_1 k_2}{m_1 m_2}}\right] * \frac{m_1}{k_2} \Delta d_1$$

(3e) * the smaller frequency is $\omega_-^2$

$$\Rightarrow d_1(0) = \bar{d}_1 + \Delta d_1 \quad \& \quad d_2(0) = \bar{d}_2 + \frac{m_1}{k_2}\left[\frac{k_1 + k_2}{2m_1} - \frac{k_2}{2m_2} + \sqrt{\left(\frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2}\right)^2 - \frac{k_1 k_2}{m_1 m_2}}\right] \Delta d_1$$

with $\Delta d_1$ arbitrary