12/5/18
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Exam #2 Solutions

(1a) $* F^{\mu\nu} F_{\mu\nu} = -2 F^{0i} F^{0i} + F^{ij} F^{ij} = -\frac{2}{c^2} \vec{E} \cdot \vec{E} + \epsilon^{ijk} B^k \epsilon^{ij\ell} B^\ell = \boxed{-\frac{2}{c^2} \vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B}}$

(1b) $* \Delta x^\mu = x_3^\mu - x_1^\mu = \begin{pmatrix} 4 \\ \vec{z} - \vec{x} \end{pmatrix} \rightarrow \Delta x^\mu \Delta x_\mu = -(4)^2 - \|\vec{z} - \vec{x}\|^2 = -14 > 0 \rightarrow$ timelike

(1c) $*$ take $\boxed{\hat{\beta} = \frac{1}{\sqrt{2}} (\vec{z} - \vec{x})}$

$$\rightarrow \Lambda^\mu{}_\nu \Delta x^\nu = \begin{pmatrix} \gamma & -\gamma \beta \hat{\beta}^n \\ -\gamma \beta \hat{\beta}^m & \delta^{mn} - \hat{\beta}^m \hat{\beta}^n (1-\gamma) \end{pmatrix} \begin{pmatrix} 4 \\ \sqrt{2} \hat{\beta}^n \end{pmatrix} = \begin{pmatrix} 4\gamma - \sqrt{2} \beta \gamma \\ \hat{\beta}^m (-4\beta + \sqrt{2}) \gamma \end{pmatrix} = \begin{pmatrix} \sqrt{14} \\ 0 \end{pmatrix} \text{ for } \boxed{\beta = \frac{\sqrt{2}}{4}}$$

(1d) $* x_1^\mu$ to $x_2^\mu \rightarrow \vec{u} = \frac{c}{2} (\vec{y} - \vec{x})$

$* x_2^\mu$ to $x_3^\mu \rightarrow \vec{v} = \frac{c}{2} (\vec{z} - \vec{y})$

$$\rightarrow \left\{ \begin{array}{l} \frac{\vec{u} \cdot \vec{v}}{c^2} = -\frac{1}{4}, \ \beta^2 = \frac{1}{2} \ \& \ \sqrt{1-\beta^2} = \frac{1}{\sqrt{2}} \\ \vec{u}_\parallel = -\frac{c}{4\sqrt{2}} (\vec{z} - \vec{y}) \ \& \ \vec{u}_\perp = \frac{c}{2} (\vec{y} - \vec{x}) + \frac{c}{4\sqrt{2}} (\vec{z} - \vec{y}) \end{array} \right\}$$

$$* \vec{u}' = \frac{1}{1 - \frac{\vec{u} \cdot \vec{v}}{c^2}} \left[\vec{u}_\perp \sqrt{1-\beta^2} + \vec{u}_\parallel - \vec{v} \right] = \boxed{\frac{4}{5} \left[\frac{1}{\sqrt{2}} \vec{u}_\perp + \vec{u}_\parallel - \vec{v} \right]}$$

(1e) $* p_1^\mu = m_1 \gamma_u \begin{pmatrix} c \\ \vec{u} \end{pmatrix}$

$* p_2^\mu = m_2 \gamma_v \begin{pmatrix} c \\ \vec{v} \end{pmatrix}$

$$\rightarrow p_1 \cdot p_2 = \gamma_u m_1 \gamma_v m_2 c^2 \left[1 - \frac{\vec{u} \cdot \vec{v}}{c^2} \right] = \boxed{\frac{5}{2} m_1 m_2 c^2}$$

(2a) $*$ This problem is almost identical to problem 1 on Asmt #10!

$* 1 = \{Q_1, P_1\} = 3q_1^2 \frac{\partial P_1}{\partial p_1}$

$* 0 = \{Q_2, P_1\} = \frac{\partial P_1}{\partial p_1} + \frac{\partial P_1}{\partial p_2}$

$$\rightarrow P_1 = \left(\frac{p_1 - p_2}{3q_1^2} \right) + f(q_1, q_2)$$

$* 0 = \{Q_1, P_2\} = 3q_1^2 \frac{\partial P_2}{\partial p_1}$

$* 1 = \{Q_2, P_2\} = \frac{\partial P_2}{\partial p_1} + \frac{\partial P_2}{\partial p_2}$

$$\rightarrow P_2 = p_2 + g(q_1, q_2)$$

$$* 0 = \{P_1, P_2\} = \frac{\partial f}{\partial q_2} - \frac{1}{3q_1^2} \left(\frac{\partial g}{\partial q_1} - \frac{\partial g}{\partial q_2} \right) \rightarrow f(q_1, q_2) = h(q_2) + \frac{1}{3q_1^2} \int_0^{q_2} dx \left(\frac{\partial g}{\partial q_1} - \frac{\partial g}{\partial x} \right) (q_1, x)$$

(2b) $* h(q_2) = 0$

$* g(q_1, q_2) = \frac{c}{b} (q_1 + q_2)^2$

$$\rightarrow \boxed{H = \frac{a}{2} P_1^2 + b P_2} \rightarrow \begin{array}{l} Q_1(t) = Q_{10} + a P_{10} t \\ Q_2(t) = Q_{20} + bt \end{array}$$

(2c) $$* q_1(t) = [Q_1(t)]^{1/3} = \left[q_{10}^3 + a \left(\frac{p_{10} - p_{20}}{3q_{10}^2} \right) t \right]^{\frac{1}{3}}$$

$$* q_2(t) = Q_2(t) - [Q_1(t)]^{\frac{1}{3}} = q_{10} + q_{20} + bt - \left[q_{10}^3 + a \left(\frac{p_{10} - p_{20}}{3q_{10}^2} \right) t \right]^{\frac{1}{3}}$$

$$* p_2(t) = P_2(t) - \frac{c}{b} Q_2^2(t) = p_{20} + \frac{c}{b} (q_{10} + q_{20})^2 - \frac{c}{b} (q_{10} + q_{20} + bt)^2$$

$$* p_1(t) = P_2 - \frac{c}{b} Q_2^2 + 3 Q_1^{2/3} P_1 = p_{20} + \frac{c}{b} (q_{10} + q_{20})^2 - \frac{c}{b} (q_{10} + q_{20} + bt)^2 + 3 \left[q_{10}^3 + a \left(\frac{p_{10} - p_{20}}{3q_{10}^2} \right) t \right]^{2/3} \left(\frac{p_{10} - p_{20}}{3q_{10}^2} \right)$$

(2d) $*$ This is identical to problem 3 of Asmt #2!

$$* L = 8m\ell^2 \cos^2(\lambda) \dot{\lambda}^2 - 2mg\ell \sin^2(\lambda) \rightarrow \boxed{H = \frac{p_\lambda^2}{32m\ell^2 \cos^2(\lambda)} + 2mg\ell \sin^2(\lambda)}$$

(2e) $* H(p_\lambda, \lambda) = E \rightarrow$ full range is $\pm \lambda(E) = \pm \sin^{-1}\left(\sqrt{\frac{E}{2mg\ell}} \right)$

$$* J(E) = 2 \int_{-\lambda(E)}^{+\lambda(E)} d\lambda \sqrt{32m\ell^2} \cos(\lambda) \sqrt{E - 2mg\ell \sin^2(\lambda)} \rightarrow \boxed{f = \frac{dE}{dJ} = \left[\sqrt{32m\ell^2} \int_{-\lambda(E)}^{+\lambda(E)} \frac{d\lambda \cos(\lambda)}{\sqrt{E - 2mg\ell \sin^2(\lambda)}} \right]^{-1} = \frac{1}{4\pi} \sqrt{\frac{g}{\ell}}}$$

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(3a) $*\; F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu(A_\nu - \cancel{\partial_\nu\theta}) - \partial_\nu(A_\mu - \cancel{\partial_\mu\theta}) = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$

$*\; (\partial_\mu + ieA'_\mu)\varphi' = [\partial_\mu + ieA_\mu - ie\partial_\mu\theta][e^{ie\theta}\varphi] = e^{ie\theta}[\partial_\mu + ieA_\mu]\varphi$

$\rightarrow (\partial_\mu + ieA'_\mu)\varphi' * (\partial^\mu - ieA'^\mu)\varphi'^* = (\partial_\mu + ieA_\mu)\varphi * (\partial^\mu - ieA^\mu)\varphi^*$

$\therefore \mathcal{L}' = \mathcal{L}$

(3b) $*\; \dfrac{\delta S}{\delta A_\mu} = \partial_\nu F^{\nu\mu} + ie\varphi\cdot(\partial^\mu - ieA^\mu)\varphi^* - ie(\partial^\mu + ieA^\mu)\varphi\cdot\varphi^* = 0$

$*\; \dfrac{\delta S}{\delta \varphi^*} = -(\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu)\varphi = 0$

$*\; \dfrac{\delta S}{\delta \varphi} = -(\partial_\mu - ieA_\mu)(\partial^\mu - ieA^\mu)\varphi^* = 0$

(3c) $*\; \mathcal{L} = \frac{1}{2c^2}\dot{A}_i\dot{A}_i + \frac{1}{c^2}\dot{\varphi}\dot{\varphi}^* - \frac14 F_{ij}F_{ij} - (\partial_i + ieA_i)\varphi\cdot(\partial_i - ieA_i)\varphi^*$

$*\; E^i = \dfrac{\partial\mathcal{L}}{\partial\dot{A}_i} = \frac{1}{c^2}\dot{A}_i$

$*\; \Pi = \dfrac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \frac{1}{c^2}\dot{\varphi}^*$

$*\; \Pi^* = \dfrac{\partial\mathcal{L}}{\partial\dot{\varphi}^*} = \frac{1}{c^2}\dot{\varphi}$

$\rightarrow \mathcal{H} = \frac12 c^2 E^i E^i + c^2\Pi^*\Pi + \frac14 F_{ij}F_{ij} + (\partial_i + ieA_i)\varphi(\partial_i - ieA_i)\varphi^*$

$H = \int d^3x\, \mathcal{H}$

$*$ the constraint is the $\mu=0$ eqn $\rightarrow -c\vec{\nabla}\cdot\vec{E} + iec(\varphi\Pi - \Pi^*\varphi^*) = 0$

(3d) $*$ Residual gauge inv + the constraint $\rightarrow \vec{\nabla}\cdot\vec{A}(t,\vec{x}) = 0$ (in the free theory)

$\boxed{\text{Dynamical Free Field Eqns}}$

always $\vec{k}\cdot\vec{\tilde{A}} = 0$

$*\; (\partial_t^2 - c^2\nabla^2)A_i(t,\vec{x}) = 0 \rightarrow A_i(t,\vec{x}) = \int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\left[\tilde{A}^T_{i0}(\vec{k})\cos(ckt) + \tilde{\dot{A}}^T_{i0}(\vec{k})\frac{\sin(ckt)}{ck}\right]$

$*\; (\partial_t^2 - c^2\nabla^2)\varphi(t,\vec{x}) = 0 \rightarrow 0 \rightarrow \varphi(t,\vec{x}) = \int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\left[\tilde{\varphi}_0(\vec{k})\cos(ckt) + \tilde{\dot{\varphi}}_0(\vec{k})\frac{\sin(ckt)}{ck}\right]$

$*\; \varphi^*(t,\vec{x}) = [\varphi(t,\vec{x})]^*$

(3e) $*\; \partial_\mu\theta = 0 \rightarrow \delta A_\mu = 0,\ \delta\varphi = ie\theta\varphi\ \&\ \delta\varphi^* = -ie\theta\varphi^*$

$*\; \delta\mathcal{L} = 0$

$*\; \dfrac{\partial\mathcal{L}}{\partial\partial_\mu\varphi} = (\partial^\mu - ieA^\mu)\varphi^*$

$*\; \dfrac{\partial\mathcal{L}}{\partial\partial_\mu\varphi^*} = (\partial^\mu + ieA^\mu)\varphi$

$\rightarrow J^\mu = -ie\varphi\cdot(\partial^\mu - ieA^\mu)\varphi^* + ie(\partial^\mu + ieA^\mu)\varphi\cdot\varphi^*$

NB $\partial_\nu F^{\nu\mu} = J^\mu$