

Assignment #5

Reading: Complete Chapter 4 and begin Chapter 5 in *Goldstein*.

Problems: Due by the start of class on Monday, 9/24/18.

- (1) Recall that a general rotation matrix can be expressed as a rotation by angle Φ about an axis \hat{n} ,

$$A_{ij} = \left[\delta_{ij} - \hat{n}_i \hat{n}_j \right] \cos(\Phi) + \epsilon_{ijk} \hat{n}_k \sin(\Phi) + \hat{n}_i \hat{n}_j .$$

Show that the time derivative of this general form obeys,

$$\left(A^T \dot{A} \right)_{ij} = \epsilon_{ijk} \left[\hat{n}_k \dot{\Phi} + \dot{\hat{n}}_k \sin(\Phi) + \epsilon_{k\ell m} \hat{n}_\ell \dot{\hat{n}}_m [1 - \cos(\Phi)] \right] .$$

You might find the following identities to be useful,

$$\hat{n} \cdot \dot{\hat{n}} = 0 \quad , \quad a_i b_j - a_j b_i = \epsilon_{ijk} (\vec{a} \times \vec{b})_k .$$

- (2) Consider a point particle of mass m which is stationary in a rotating frame with constant angular velocity $\vec{\omega}$.
- (a) Show that the centrifugal force on the particle is minus the gradient of the centrifugal potential $V_c = -\frac{1}{2}m\|\vec{\omega} \times \vec{r}\|^2$.
- (b) If the particle is on the surface of the Earth it experiences the gravitational potential energy of the Earth, which has a small quadrupole deformation,

$$V_g = -\frac{mM_E G}{r} \left\{ 1 - J_2 \left[\frac{3}{2}(\hat{\omega} \cdot \hat{r})^2 - \frac{1}{2} \right] \left(\frac{R_E}{r} \right)^2 \right\} .$$

Here $J_2 \simeq .001083$, R_E is the Earth's equatorial radius and M_E is the Earth's mass. Water follows a surface of constant total potential energy which is known as the *geoid*. Solve for the geoid's radius at general polar angle, $R(\theta)$, to first order in the small parameters J_2 and $(\omega^2 R_E^3)/(GM_E)$ and compute the numerical value of the difference of $R(\theta)$ from its largest to its smallest values.

- (c) Derive a first-order expression for the acceleration, $g(\theta)$, due to the combined potential, $V_g + V_c$ and compute the numerical value of its largest variation.
- (3) Consider a symmetrical top consisting of a solid cone of radius r and height a , joined on its circular face to a solid cylinder of radius r and height b . Both the cone and the cylinder have the same uniform mass density ρ_0 .
- (a) What is the top's mass?
- (b) What is the top's center of mass assuming its axis of rotation is along the $+\hat{z}$ axis?
- (c) What is the moment of inertia tensor I_{ij} taking the apex of the cone to be the origin?
- (d) What are the eigenvalues and eigenvectors of I_{ij} ?