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R.P. Woodard

Corrected Lecture Notes

* Consider N particles of mass m_i & position $\vec{r}_i$ obeying k constraints

$\boxed{\text{Identities involving } \vec{r}_i = \vec{r}_i(q_1, \dots, q_{3N-k}, t)}$

* $\dfrac{d\vec{r}_i}{dt} = \displaystyle\sum_{\alpha=1}^{3N-k} \frac{\partial \vec{r}_i}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial \vec{r}_i}{\partial t} \Rightarrow \frac{\partial}{\partial \dot{q}_\alpha} \frac{d\vec{r}_i}{dt} = \frac{\partial \vec{r}_i}{\partial q_\alpha}$

* $\dfrac{d}{dt}\left(\dfrac{\partial \vec{r}_i}{\partial q_\alpha}\right) = \displaystyle\sum_{\beta=1}^{3N-k} \frac{\partial^2 \vec{r}_i}{\partial q_\alpha \partial q_\beta} \dot{q}_\beta + \frac{\partial^2 \vec{r}_i}{\partial t \partial q_\alpha} = \frac{\partial}{\partial q_\alpha}\left[\sum_{\beta=1}^{3N-k} \frac{\partial \vec{r}_i}{\partial q_\beta} \dot{q}_\beta + \frac{\partial \vec{r}_i}{\partial t}\right] = \frac{\partial}{\partial q_\alpha} \frac{d\vec{r}_i}{dt}$

* $\delta \vec{r}_i = \displaystyle\sum_{\alpha=1}^{3N-k} \frac{\partial \vec{r}_i}{\partial q_\alpha} \delta q_\alpha$

$\boxed{\text{No Virtual Work}}$

unknown forces of constraint

* $m_i \dfrac{d^2\vec{r}_i}{dt^2} = \vec{F}_i^{(a)} + \vec{f}_i \Rightarrow \displaystyle\sum_{i=1}^{N} \delta\vec{r}_i \left[m_i \frac{d^2\vec{r}_i}{dt^2} - \vec{F}_i\right] = \sum_{i=1}^{N} \delta\vec{r}_i \cdot \vec{f}_i = 0$

$\boxed{\text{Conservative Force Term}}$

* $\vec{F}_i = -\vec{\nabla}_i V(\vec{r}_1, \dots, \vec{r}_N)$

* $-\displaystyle\sum_{i=1}^{N} \delta\vec{r}_i \cdot \vec{F}_i = +\sum_{\alpha=1}^{3N-k} \delta q_\alpha \sum_{i=1}^{N} \frac{\partial \vec{r}_i}{\partial q_\alpha} \cdot \vec{\nabla}_i V = \sum_{\alpha=1}^{3N-k} \delta q_\alpha \frac{\partial V}{\partial q_\alpha}$

$\boxed{\text{Kinetic Energy}}$

* $T \equiv \dfrac{1}{2} \displaystyle\sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt} \cdot \frac{d\vec{r}_i}{dt}$

* $\dfrac{\partial T}{\partial \dot{q}_\alpha} = \displaystyle\sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt} \cdot \frac{\partial}{\partial \dot{q}_\alpha} \frac{d\vec{r}_i}{dt} = \sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt} \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} \equiv \text{Identity } 1$

Kinetic Energy (Continued)

* $\frac{\partial T}{\partial q_\alpha} = \sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt} \cdot \frac{\partial}{\partial q_\alpha} \frac{d\vec{r}_i}{dt} = \sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt} \cdot \frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_\alpha}$ $\equiv$ Identity ②

Kinetic Term

* $$\sum_{i=1}^{N} m_i \frac{d^2\vec{r}_i}{dt^2} \cdot \delta\vec{r}_i = \sum_{\alpha=1}^{3N-k} \delta q_\alpha \sum_{i=1}^{N} m_i \frac{d^2\vec{r}_i}{dt^2} \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha}$$

$$= \sum_{\alpha=1}^{3N-k} \delta q_\alpha \left\{ \frac{d}{dt} \Bigg[\underbrace{\sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt} \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha}}_{\text{use Identity ①}} \Bigg] - \underbrace{\sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt} \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_\alpha}\right)}_{\text{use Identity ②}} \right\}$$

$$= \sum_{\alpha=1}^{3N-k} \delta q_\alpha \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} \right\}$$

Final Point

* $0 = \sum_{\alpha=1}^{3N-k} \delta q_\alpha \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} + \frac{\partial V}{\partial q_\alpha} \right\}$ true $\forall\, \delta q_\alpha$ $\Rightarrow$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$

where $L = T - V$

* NB this is even true for velocity-dependant forces